TEACHING AND LEARNING GEOMETRY AND BEYOND…

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Geometry and its teaching have always been a problematic and exemplar issue regardless of the period. Torn between utilitarian and idealistic visions, the very nature of geometry has moved within very wide margins from regarding it as sacred to aiming at its disappearance. Regarding mathematics education, researches on geometry have raised the attention of many prominent researchers in the domain such as for instance Freudenthal and Brousseau. What are today the core items and the contributions of researches in the didactics of geometry, a domain in which the current development of specific software has caused quick changes? We address this question in the light of the results of recent researches and also the rich discussions which have been occurring in the CERME Working Group on geometry from its beginning in 1999. We also develop some ideas about the perspective of geometric paradigms and spaces for geometric work (SGW) and show how it allows describe and change the nature of geometric activity in various educational contexts.

First of all, I would like to thank the organizers and the members of the scientific Committee for their invitation to give this plenary on geometry teaching and learning.

During this talk, I present some possible orientations for researches within the field of geometry didactics. The main point is, to me, that we should take advantage to focus on what I call Geometric work to advance and to develop new views on geometry education. And I develop, with some details, this idea and the framework related to it during the presentation. Another point is that we would get some benefits by linking geometry to other maths areas and to technological tools. That explains partially the meaning of Beyond in the title.

WHY TEACH AND LEARN GEOMETRY TODAY?

For a long time mathematics has been synonymous with geometry and questioning the usefulness of mathematics was equivalent to questioning geometry. Today, it is somehow different but we can learn from the past in order to think on the question but keeping in mind how the current situation is specific.

In *An essay on the usefulness of mathematical learning* written in 1701, Arbuthnot, an English physician, tried to persuade the rich people of his time to learn and practice mathematics. He based his argumentation on three points which are always interesting to consider:

1. Develop Mind and Reasoning. “Truth is the same thing to the Understanding as Music to the Ear and Beauty to the Eye”, he wrote in the flourishing style of his time. This argument is classic and will be used and summarized later with the famous “For the honour of human spirit” of Jacobi (1830) quoted by Dieudonné (1987).

2. For their applications in a wide variety of fields. Arbuthnot favoured Trade, Navigation, Art of War...

3. To learn how to get to the results and not only the results. This means that the path is as important as the result. Arbuthnot praised mathematics and geometry as a method of freeing the mind from superstition.

The third argument keeps its value today in a world with a lot of technologies meaningless for common people and at the same time an increasing strength of superstitions.

Nearer from to-day and into the field of maths education, the “modern maths” revolution and the subsequent counter-reform have questioned geometry through the name of Euclid. “A bas Euclide” was Dieudonné’s provoking motto against traditional geometry based on an amount of triangle properties disconnected from the evolution of contemporary science. In a same way but for different reasons, teachers and researchers involved in the counter-reform rejected Euclid because he did not give any efficient method to apply it in the real world problems. Another marginal and provoking view was Brousseau's idea of considering Euclid as the first didactician. Indeed, Euclid wrote a text organizing knowledge and used, with some adaptations, as a base for textbooks during centuries and up to the beginning of the XXth century.

To view the variety of points of view, eventually conflicting, it is interesting to quote this remark by Fletcher, a well-known math educator, in an ESM special issue on geometry published in 1971

> The cry "Euclid must go!" has gained a certain notoriety in recent years. Our reaction to this in England was merely mild surprise since as far as we were concerned Euclid had already been gone for a long time. (Fletcher, ESM. 3-3, 1971).

This remark shows how the teaching traditions and the relationships with geometry are different among countries which may be geographically very close.

Nowadays, all these questions and conflicting viewpoints coexist and the teaching and learning of geometry have been to be developed in a changing context characterized by the tension between utilitarian and idealist visions on mathematics with an advantage to the utilitarian approaches. At the same time the use and potentialities of Dynamic Geometry Software (DGS) have deeply changed the way of discovering and proving in the domain and created a new relationship with Truth and Proof within maths education.

To progress in the direction of making our knowledge grow on how and what to teach and learn in Geometry, researchers in the domain can use the great amount of texts elaborated for the group on geometry which was existing in the Conferences of ERME since the first Conference. Among the numerous papers presented in the working group on Geometry, we can distinguish some recurrent and relevant points:

1. Development of spatial abilities and geometrical thinking through consecutive educational levels.

2. Geometry education and the ``real world'': geometrisation and applications

3. Instrumentation: artefacts such as, computers and the way they are used

4. Explanation, argumentation and proof in geometry education.

To this four classic topics in the domain we can add some theoretical aspects which in a certain sense are local and specific to the domain: Van Hiele's levels; Duval's registers of semiotic representation; Houdement and Kuzniak's geometrical paradigms.

The need for a common framework related to Geometry education appeared necessary in the working group in order to facilitate exchanges among members and to allow a capitalisation of knowledge in the domain. Due to collaborations initiated during Cerme meetings with colleagues from Cyprus, Spain and Canada or other from Mexico and Chile, it has been made possible to develop a theoretical framework that I will introduce. In our mind, the framework should be dedicated to study the teaching and learning of elementary geometry on the whole educational system that means during compulsory education and also teacher training. It should be neutral in the sense that is can be used to compare the teaching of geometry in different countries and institutions without any a priori on "best" directions. For that it appeared very soon, that it could be interesting to focus on the nature and form of the effective geometric work made by students and teachers in Geometry.

MATHEMATICAL WORK CONSIDERED A CRUCIAL POINT

As it has been underlined above, the notion of geometric work is central in the approach and we start by detailing what is geometric work for us. First we need to precise, more generally, our view, oriented by educational perspectives, on mathematical work.

In the special issue of ESM already quoted, Freudenthal (1971) found it useful to answer to the question "What is mathematics" before presenting his ideas on geometry education. Addressed to teachers and researchers considered as mathematicians, he put the stress on two aspects of the work in the domain: the activity of solving problems and the activity of organizing.

> Of course you know that mathematics is an activity because you are active mathematicians. It is an activity of solving problems, of looking for problems, but it is also an activity of organizing a subject matter. ESM 3-3 – 1971

In a same vein but a step further, the well-known conception of Thurston (1995), Fields medal in 1982, give a shared view on mathematics considered a human activity.

> Mathematics includes integers numbers and geometry plane and solids
>
> Mathematics is what Mathematicians study
>
> Mathematicians are those human beings who make advance human understanding of mathematics.

At a first glance, the definition looks circular, but it is not. Initiated on numbers and geometry plane and solids, it creates a dynamic between mathematics knowledge and people who make mathematics. Both aspects are important in this work which relates intimately epistemological and cognitive aspects through the image of a mathematician that we can consider a cognitive subject in charge of the "human understanding of mathematics".

Once mathematics is clearly defined as a human activity, it is easy to turn to the idea of mathematical work including and orienting these activities, but it remains to characterise the specificity of such work and for that, Habermas's (1985) consideration on work defined as a rational activity oriented toward an end will be useful.

> By work or rational activity relative to an end, I hear or an instrumental activity, or a rational choice, or else a combination of both.

Boero developed during his conference some aspects of what rationality is for Habermas, and I will not insist on this point but only retain, for our framework related to education, the necessity of thinking mathematical work as a rational human activity oriented toward a better understanding of specific topics.

How can we interpret and use this in Geometry education? Again, Freudenthal in his paper warns again the taste of mathematicians and educators to restrain mathematics work to organizing.

> A great part of mathematical activity today is organizing. We like to offer the results of our mathematical activity in a well organized form where no traces betray the activity by which they were created. This objectivation is a habit of mathematicians from the oldest times. It is a good habit, and it is a bad one. We freeze up the result of our activity into a rigid system, because this is objective, because it is rational, and because it is beautiful,and this we teach.

To avoid the risk of freezing up the results of mathematic work, it will be necessary to introduce the idea that several work context exist. Two of them are classically identified: a context of discovery where new results and solutions of problems are sought; a context of justification where discoveries are proved and presented to a larger community with its proper rules and style of work. We can add a context of use where the results become familiar to the user, are applied to solve problems which are not necessary mathematical. This variety of contexts need to be kept in mind when developing activities within an educational system and it implies various forms and phases of student's work: researching, presenting, practising...

GEOMETRIC WORK AND ITS SPACE

To study specifically the geometric work within the scope of education, we have introduced the idea of a space, named Space for Geometric Work (SGW), organized to ensure the work of people solving geometrical problems. The subject may be an ideal expert (the mathematician) or a student or senior student in mathematics. Problems are no part of the space but they justify it and speed up its construction.

Initially, this idea was suggested by architects' definition of work spaces as places to be built to ensure the best practice of a specific work (Lautier, 1999).

To ground SGW, we will think of it through epistemological and cognitive dimensions which structure the whole work. As we noted before, the former is in charge of the coherence of the mathematics content and the latter refers to the cognitive subject supposed to solve geometric problems.

According to the epistemological dimension, we introduced three characteristic components of the geometrical activity in its purely mathematical dimension. These three interacting components are the following ones:

A real and local space as material support with a set of concrete and tangible objects.

A set of artefacts such as drawing instruments or software.

A theoretical reference frame based on definitions and properties.

These components are not simply juxtaposed, they must be organized with a precise goal depending on the mathematical domain in its epistemological dimension. This justifies the name of epistemological plane given to this level. From the point of view of geometry considered as a mathematical theory, the theoretical frame of reference is crucial, even if for the users it is sometimes implicit or hidden.

From Duval (1995), we have adapted the idea of three cognitive processes involved in geometrical activity and structuring the cognitive level.

A visualization process connected to the representation of space and material support;

A construction process determined by instruments (ruler, compasses, etc.) and geometrical configurations;

A discursive process conveying argumentation and proofs.

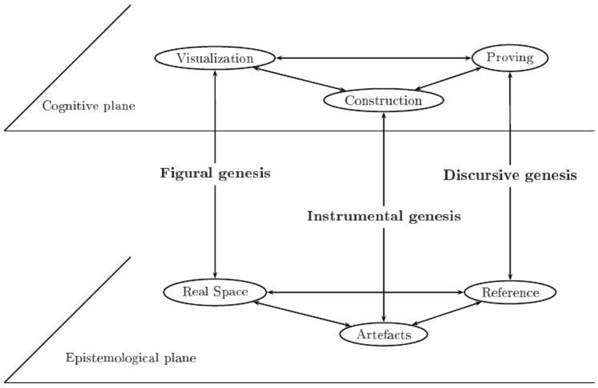
In our approach, both levels, cognitive and epistemological, need to be articulated in order to ensure a coherent and complete geometric work. This process supposes some transformations which can be pinpointed through different ways. It is possible to refer to general notion like intuition, experiment and deduction as Gonseth (1952) did in his major book on geometry. But here, in order to insist on the developmental process involved in the constitution of SGW, the notion of genesis has been used. For us, a genesis involves the development and not only the origin of a process. Strictly related to our ternary conception of each level, three genesis need to be considered:

An instrumental genesis which transforms artefacts in tools within the construction process.

A figural and semiotic genesis which provides the tangible objects their status of operating mathematical objects.

A discursive genesis of proof which gives a meaning to the properties used within mathematical reasoning.

This can be summarized and illustrated by a diagram which will play at the time a metaphoric role and be a prospective tool to think about SGW.



Space for Geometric Work and its geneses

The representation of each level by a plane does not mean that these levels are strictly plane and parallel, and the distance between the poles – the length of the processes necessary to articulate the two levels – depends on and differs from one pole to the other regarding the problem and the tools used. On the other hand, as arrows appear in the diagram, it will be necessary to see what could be the meaning of each way when we want to describe the effective work made when solving a problem.

WHAT GUIDES THE WORK? IN SEARCH OF GEOMETRIC PARADIGMS

We will start to answer the question on what guides the geometric work by giving a first example (Kuzniak and Rauscher, 2011) which among numerous others of the same kind shows that a single viewpoint on geometry would miss the complexity of the geometric work, due to different meanings that depend both on the evolution of mathematics and school institutions.

> Let ABC be a triangle with a right angle in B, with AB=4 cm and BC=2 cm. The ray (Ax) is perpendicular to the line (AB). And M is a point on the ray (Ax). The purpose of this problem is to obtain particular configurations of the triangle AMC.
>
> Question: Does a point M exist such that the triangle ACM is equilateral? Justify your answer.

This problem was given to a lot of students at different grades but especially to pre-service teachers. In this case, students have a high general and university level and no problem with reasoning and formulating an answer.

A common answer which appeared was the following:

> The correct answer is “no” and it can be shown, using compasses, that there is no third vertex on the ray (Ax) for an equilateral triangle constructed on the side (AC).

Such a response is emblematic of what we named Geometry I. The student carries out an experiment in the real, perceptible world by constructing a triangle with drawing instruments and then s/he realises that no crossing points lie on the line where they should be for the triangle to be equilateral. The argument is supported by diagrams, objects that are typical and central to Geometry I.

This response, however, does not match what is expected in French traditional education at this level. A solution without any measurement and information supported by the drawing is ruled out. And for a student, it is better to propose this kind of solutions:

> If ACM is an equilateral triangle with M on Ax, the angle MAC will measure 60° and the angle CAB 30° (sum of the three angles of a triangle) and by symmetry ∠CAC' will be 60° (C' is the symmetric of C through the line (AB)).
>
> As the triangle CAC' is isosceles in A (by symmetry), it should be equilateral. This is not true because the length of C'C is 4, which is unequal to CA and C'A (2.sqrt(5) by Pythagoras' theorem).

This solution is illustrative of Geometry II. A reasoned deductive argument is constructed on the basis of initial data and geometric theorems.

From this example, it cannot be induced that deduction does not exist in Geometry I as we can see it with the following solution:

> We can explain this by the fact that in an equilateral triangle all the angles are equals and the sum of the angles is 180°. The value of each is 60° . In this case, when we measure with a protractor, we observe that CAM is more than 60°, indeed CAM = 64◦.

This student deduced some properties belonging necessarily to the figure and then he checked directly on the drawing that the property is not true.

The notion of geometrical paradigm is useful for understanding, clarifying and organising the various and conflicting points of view observed in education. In our framework, we use the notion of paradigm according to Kuhn's definition. In his fundamental book about scientific revolution, Kuhn (1966) uses this term many times and after some approximations, he defines it by putting the stress on two aspects.

> In its most global use, the term paradigm stands for the entire constellation of beliefs, values, techniques, practices etc. shared by the members of a given community.
>
> On the other, it denotes one sort of element in that constellation, the concrete problem-solutions which, employed as models or examples, can replace explicit rules as basis for the solution of the remaining problems of normal science.

The concept of paradigm broadens the notion of theory and relates it to the existence of a community of individuals who share a common theory.

A paradigm is what the members of a scientific community share, and, a scientific community consists of men who share a paradigm (Kuhn, 1966).

Interpreted in the education world, it gives sense to the question about students' and teachers' different work in problem solving. We can argue that they are working in distinct paradigms and this epistemological difference can explain some didactic misunderstandings.

THREE ELEMENTARY GEOMETRIES

Geometrical paradigms were introduced into the field of didactics of geometry to take into account the diversity of points of view (Kuzniak and Houdement, 1999, 2003) and we summarize our findings by quoting former papers and especially (Kuzniak and Rauscher 2011, Kuzniak 2011).

To bring out geometrical paradigms, we used three viewpoints: epistemological, historical and didactical. That led us to consider the three following paradigms described below.

Geometry I: Natural Geometry

Natural Geometry has the real and sensible world as a source of validation. In this Geometry, an assertion is supported using arguments based upon experiment and deduction. Little distinction is made between model and reality and any arguments are allowed to justify an assertion and convince others of its correctness. Assertions are proven by moving back and forth between the model and the real: The most important thing is to develop convincing arguments. Proofs could lean on drawings or observations made with common measurement and drawing tools such as rulers, compasses and protractors. Folding or cutting the drawing to obtain visual proofs is also allowed. The development of this geometry was historically motivated by practical problems.

The perspective of Geometry I is of a technological nature.

Geometry II: Natural Axiomatic Geometry

Geometry II, whose archetype is the classic Euclidean Geometry, is built on a model that approaches reality. Once the axioms are set up, proofs have to be developed within the system of axioms to be valid. The system of axioms could be incomplete and partial: The axiomatic process is a work in progress with modelling as its perspective. In this geometry, objects such as figures exist only by their definition even if this definition is often based on some characteristics of real and existing objects.

Both Geometries are closely linked to real world even if it is in various ways.

Geometry III: Formal Axiomatic Geometry

To these first two approaches, it is necessary to add a third Geometry (Formal Axiomatic Geometry) which is little present in compulsory schooling but which is the implicit reference of teachers' trainers when they have studied mathematics in university, which is very influenced by this formal and logical approach. In Geometry III, the system of axioms itself, disconnected from reality, is central. The system of axioms is complete and unconcerned with any possible applications to the world. It is more concerned with logical problems and tends to complete "intuitive" axioms without any "call in" to perceptive evidence such as convexity or betweenness.

Moreover, axioms are organized in families which structure geometrical properties: affine, euclidean, projective, etc.

These three approaches (and this is one original aspect of our viewpoint) are not ranked: Their perspectives are different and so the nature and the handling of problems change from one to another. More than the name, what is important here is the idea of three different approaches of geometry: Geometry I, II and III. It must also be clear that Geometry I is not a poor and small geometry for young students even if it is the first that they encounter. Abstract and complex forms of this geometry exist as it can be found in Lemoine's work on geometrography or Klein's students researches on approximation made by industrial drawing makers and using probabilistic theory to estimate the effects of errors.

Various SGW

A SGW exists only through its users, current or potential. Its constitution depends on the way users combine the two planes and their components for solving geometric problems. It also depends on the cognitive abilities of a particular user, expert or beginner. The make-up of a GWS will vary with the education system (the reference GWS), the school circumstances (the implemented or suitable GWS) and on the practitioners (students' and teachers' personal GWS). In practice, the constitution of a GWS does not rely on a single paradigm, but rather on the interplay among different paradigms and a specific study of each level is necessary. Before giving some examples we will detail these various GWS involved in Geometry education and relate them to different kinds of vigilance: epistemological, didactic and cognitive.

The reference SGW or the expected reorganization This space is normally defined and based on mathematical criteria. But it also depends on social, economical and political criteria. Studies of treatises written by mathematicians or maths educators and of the intended curriculum will allow describe this level in which an epistemological vigilance is at stake. This means that the rules of functioning of this SGW do allow knowledge to be organized in a well-defined and coherent domain.

At this point of the curriculum, the good functioning of the personal SGW is the ultimate goal of geometry teaching and learning and this point needs a cognitive vigilance. Here, the cognitive plane is concerned by a specific individual and not an epistemic or institutional subject, and to know more about its contents, conceptions, knowledge of students have to be studied through problem solving and questionnaires.

Between the two planes, and fundamental in the make-up of a coherent and global geometric work, it remains to focus on the implemented SGW concerned by the didactic vigilance which will assure that the personal student's work corresponds to what the reference SGW proposes. Indeed, when a general paradigm is accepted and the reference SGW built, it remains to teach geometry to students and for that it's necessary to organize a suitable SGW to convey the kind of geometry expected by the educational institution. The geometrical working space turns to be suitable only if it allows the user link and master the three components defining the working space.

Curriculum, textbooks, observation of real class implementation and preparation will support the study.

AN EXAMPLE OF COHERENT AND QUASI ASSUMED GEOMETRY I

To show what a suitable and implemented SGW fitted to Geometry I could be, an example will be given, taken from a comparative study of the teaching of geometry in France and Chile (Guzman and Kuzniak 2006).

Following a standard model in Hispanic world, education in Chile is divided into elementary school (Básica) till Grade 8 and secondary school (Media) till Grade 12. From 1998 on, the teaching of mathematics has left aside the very abstract teaching which was in place before and turned into a more concrete and empirical way. And today, the reference SGW is underlined by Geometry I. To illustrate this and point out some differences between France and Chile, let us observe the following exercice taken from the textbook *Marenostrum* (Grade 10).

The problem is given to students starting the chapter on similarity and the solution will be given later in the same chapter:

> Alfonso is just coming from a journey in the precordillera where he saw a field with a quadrilateral shape which interested his family. We want to estimate its area. For that, during his journey, he measured, successively, the four sides of the field and he found approximately : 300 m, 900 m, 610 m, 440 m. Yet, he does not come to find the area.
>
> Working with your classmates, could you help Alfonso to determine the area of the field?

The exercise is then completed by the following hint:

> We can tell you that, when you were working, Alfonso explained its problem to his friend Rayen and she asked him to take another length of the field: a diagonal.
>
> Alfonso has come back with the datum: 630 m.
>
> Has it done right? Could we help him now, though we could not do it before?

The proof suggested in the book begins with a classical decomposition of the figure in triangles based on the indications given by the authors. But the more surprising for a French reader is to come: the authors ask to measure the missing height directly on the drawing . We recall that this way of doing is strictly forbidden at the same level of education in France.

> How can we compute the area now?
>
> Well, we determine the scale of the drawing, we measure the indicated height and we obtain the area of each triangle (by multiplying each length of a base by the half of the corresponding height).

In this case, the geometrical work is clearly within Geometry I and goes back and forth between the real world and a drawing which is a schema of the reality. Measuring on the drawing gives the missing data. The activity is logically ended by a work on the approximation closely related to a Geometry based on the possibility of measuring.

THE IMPACT OF THE SOFTWARE ON THE IMPLEMENTED FRENCH SPACES FOR GEOMETRIC WORK

In her master dissertation, Boclé (2008) described the typical situation given in French textbooks to introduce a new notion in geometry at the end of junior high school. In textbooks conceived just after 1996, the typical structure [SP1] was the following:

1. Construction of some particular figures with drawing instruments.
2. Measurement on these figures by using instruments (marked ruler or protractor).
3. Conjecture of a property.
4. Institutionalization of the property, either accepted without proof or formally proved later.

In the textbooks printed after 2005, a new tendency appears. A new notion is introduced using dynamic geometry software (DGS). The typical situation [SP2] is then the following one:

1. Construct a figure with DGS..
2. Get measures from the software.
3. Drag points to notice that the property remains true.
4. Institutionalize the property, either accepted or accepted without proof or formally proved later.

In both cases, to introduce the property, students have to construct several figures satisfying some criteria. Thanks to the measures made on the figures, it is possible to notice an invariant then to make a conjecture. In the textbooks following the 2005 syllabus, the activities of construction and measuring imply the use of DGS. At its beginning, every activity is clearly in SGW directed by Geometry I and favouring perception and instrumentation. In both approaches, with and without software, the point 4 is crucial for determining the type of geometry really used and the appropriate SGW. If the property is only proved in a deductive way without any use of measuring, it is possible to enter Geometry II. But what happens if the property is not demonstrated? It seems that students remain in Geometry I.

These typical situations fulfil well the curricular instructions recommending the implementation of activities leading to conjecture properties. The recent emphasis on the use of DGS is taken into account in textbooks but the real contribution of the software in the transition from Geometry I to Geometry II deserves to be questioned. Indeed, the use of a DGS is justified in the textbooks by improving the measuring accuracy and the possibility of multiplying the examples. But a measure remains an approximation and therefore is not exact.

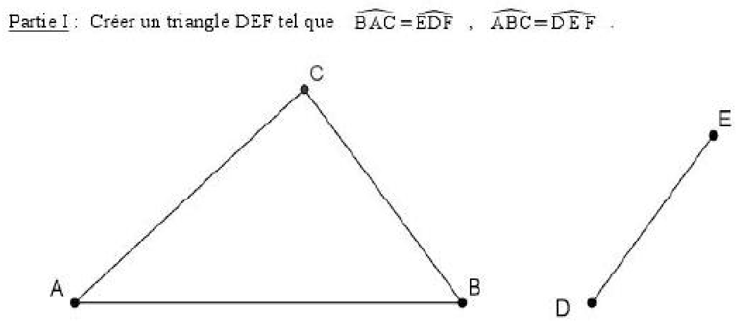
This vagueness can create a contradiction in the classroom and lead some students to become convinced by another way and then been led to prove without any measurement. By contrast, insisting on the precision of the software and its advantage with regard to ruler-and-compasses constructions risks to turn away students from the necessity of proving, which was one of the stakes expected within the reference SGW.

In her work, Boclé tried to see if the use of software in these typical situations favoured the transition to Geometry II or either if on the contrary it created a blocking element. She noticed that the strength of the proof by experiment overcame the classic work on demonstration with a purely deductive proof. In that case, it seems that the use of the software in standard situation stabilizes rather a SGW of Geometry I type and not a transition toward GII.

THE BREAK ACHIEVED IN GRADE 10 OR WHEN OSTENSION BECOMES DEMONSTRATION.

This contradiction is to be found again between the work expected by the institution and the work effectively set up in the teaching of similar triangles in an ordinary class at Grade 10. Similar triangles were reintroduced in French compulsory education in 2000. The notion has been removed from the syllabus since the modern maths reform and it reappeared in a quite different context in 2000 at Grade 10. Similar triangles are not considered by the programs as a new notion but as an opportunity to stabilize the geometric work at the end of compulsory education. We shall consider here only the result of a session managed by a teacher who first follows the typical way [SP1] but who changes on phase 4 (institutionalization) and then follows the process [SP2] by using the software by huimself.

The activity is the first one about similar triangles. A sheet of paper is given to the students with a drawing on and the first task is to create a triangle DEF such that $\angle BAC=\angle EDF$, $\angle ABC=\angle DEF$.



Below the figure, the following questions appear on the sheet given to the students:

What can we say about $\angle ACB$ and $\angle DFE$?

Compare the sides of the triangles with your ruler. What can be noticed?

Complete the sentence : We can guess that if two triangles have …. then their sides are ….

For the teacher the construction is not a problem. He anticipated two possible configurations, which seems an interesting difficulty to him. He wants to motivate in Geometry I the origin of a property which will belong completely to Geometry II when it will have been proved in the following lesson. For him, the figure is a generic example and he has not really thought about the measures given on the sheet.

The great majority of students, but not all, undertake completely the activity of construction which turns out to be long and complex. Students have difficulties with the use of their drawing instruments: the task « to make an equal angle » does not fit a well-known technique. Furthermore, the two possibilities for the final figure cause problems in the class since students are working on particular and not on general figures.

Other students understood that the construction is not important for the teacher and they quietly wait that the course goes on. They give, by abduction, purely linguistic conjectures by trying to adapt their mathematical knowledge to the situation. At the same time, students engaged in the construction task produce very different and contradictory results but actually these results and the work of these students will be left aside by the teacher who will favour the solution with DGS (Geogebra) and present it by video-projection in the class. The teacher follows the SP2 structure but without making any devolution to the students. He is the unique user of the software and he makes an institutionalization denying all the previous work of the students.

On the computer, the figure is the starting point and measures are given with five digits, even for angles. The proportion ratio calculated by the computer was 1.875 and was exactly the same for the three ratios.

The accuracy of the measures given by the computer shows to students the imperfection of their work with instruments on a very violent way. Strictly speaking, the students' work is rather useless because it is left aside by the teacher. Moreover, the accuracy of the software turns it into a tool for proof and a source of truth and, this, without the teacher knowing, as it can be seen in the following dialogue, which closes the class after the statement of the conjecture.

Teacher: Did we demonstrate the property?

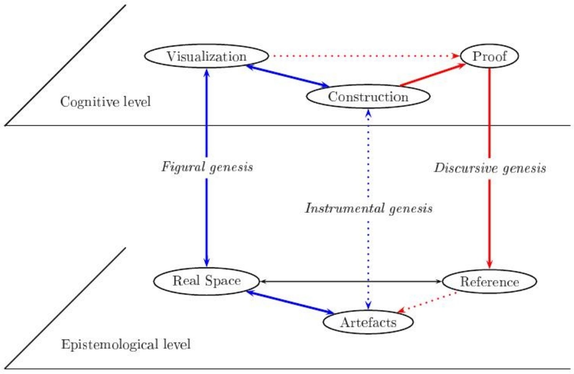
Almost all the students: Yes! We have done a demonstration.

Teacher (taken aback): Hum… No, it is too imprecise!

So after more than three years of progressive entrance in Geometry II and despite the programs which insist on the necessary awareness of the status of the statements, accepted or demonstrated, the gap between the expected work and the effective work is deep. It largely results from the appropriate SGW proposed to the students being itself very ambiguous and probably fundamentally a surreptitious Geometry I.

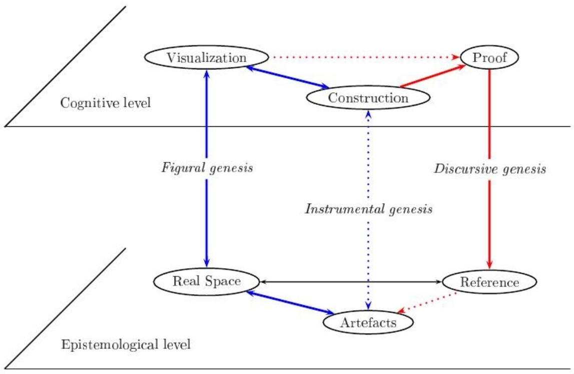
Within the SGW framework, it is possible to follow the break between two approaches of Geometry through various diagrams

For the teacher, the construction is simple and will not cause trouble. His idea it to motivate the entrance in Geometry II by a preliminary work in Geometry I (in blue) and then to introduce a formal proof based on properties to justify the construction (in red).



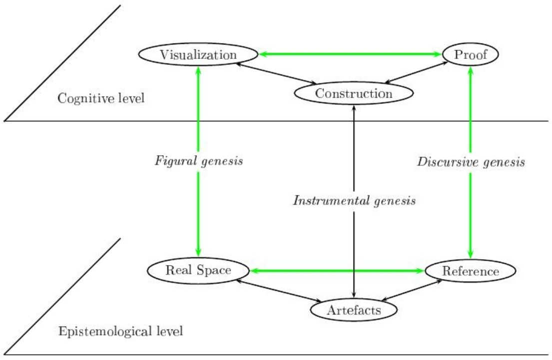
Teacher's expectations

Students with low level in geometry try to construct with drawing tools. This is complex and requires a long time because the expected construction is not based on a standard technique at this grade. Moreover, due to a diversity of measurement results, a great diversity of properties are drawn by the students who consider the drawing as a particular figure without any general nature.



Students' geometric work

The construction work made by students is ignored by the teacher who gives the solution using a software, he is the only user of this tool which is really different from classical drawing tools, both in its uses and the precision of measurements. So after the “monstration” on a screen, students immediately conclude from results based on construction (in red) and the software appears as a source of Truth, but grounded on experiment and no more on pure reasoning as in Arbuthnot's approach.



Students' geometric work without instruments

The last group of students give the conclusion without any construction work and can be summarized with the green diagram which shows an incomplete geometric work. They understand that the construction with drawing tools will be of no use for the conclusion and they complete the questions by abduction.

Even if the reference SGW always insists on a transition to GII based on GI, the implemented SGW is unstable and depends on the level of the students. For most of them, a real shift to Geometry I is favoured by the software which gives the "proof". Due to a lack of a theoretic system to ground the teaching, the reorganisation of the SGW is led only by the teacher who attempts to adapt his teaching to the supposed low level of his students. In short, a lack of epistemological vigilance conducts to a lost of cognitive vigilance.

CONCLUSION

The forms for teaching geometry and its need have always been questioned and discussed. But today, the traditional view of a geometry education useful for training logical reasoning is reconsidered by our society increasingly technological and consumerist.

With geometrical paradigms, it is possible to make explicit different stakes involved in the teaching of geometry. Each paradigm stresses a different view of "mathematical culture" considered, on the one hand, essentially practical with applications to real world or, on the other hand, more theoretical and guided by internal mathematical requirements. From this, it results that geometric work depends on various factors which can be described by the concept of SGW.

But again, the reference SGW remains difficult to determine in some countries. In France, today, the debate between the two lines is not over, even if the utilitarian line is strengthened by international institutions and evaluations like PISA which promote an empirical and utilitarian view on mathematics. So, the emergence of the suitable SGW looks very confuse and that explains partly the fickleness of personal SGWs which do not seem to reach a stable point and depend a lot on the didactical contract.

In other countries the choice of Geometry I is clearly assumed during the major part of education with a sudden change at the end of the syllabus toward a traditional SGW of type (GII/gI) with ancient forms of teaching based on Euclidean tradition.

Whatever is the chosen paradigm, a consistent geometric work has to be implemented into the classroom in order that students can solve interesting problems and are aware of the authorized tools to justify their results. So, from a Geometry I perspective, a work on approximation is essential to fix the degree of confidence to be given to the results. Similarly, in Geometry II, students and teachers need to have clear ideas on the role and use of properties. One route to be explored could be to make explicit the existence of two geometries and to seek solutions based on GI or GII for some specific problems especially in the case of modelling. Another approach is needed to overcome the conflicting viewpoints on proof and it would be interesting to relate geometric activity to other mathematical areas. Changes of areas, changes of semiotic representations have always been at the centre of mathematical work. In the case of geometry, solutions of problems are based on different sets of numbers, use of functions and all this is reinforced by new software with great semiotic potentialities. By focussing on three main geneses, semiotic – instrumental – discursive, SGW provides a framework suitable to take into account the main key points of individual mathematical work which must be linked one with another to develop a global and effective work.

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