Using the geometric working spaces to plan a coherent teaching of geometry

Alain Kuzniak and Assia Nechache

Laboratoire de Didactique André Revuz, University Paris Diderot, Paris, France

kuzniak@math.univ-paris-diderot.fr and assia.nechache@hotmail.fr

A frequently-asked question by in-service teachers during training sessions relates to the overall organisation of the geometric work at primary school level. Indeed, teachers have difficulties to integrate the various resources and activities discovered during their training in a general progression for their teaching of geometry. This issue is difficult even for teacher trainers who have to meet students' expectations. In this paper, we consider some elements that could help teacher trainers devise an overall coherent teaching of geometry. We address this issue from the analysis of a teaching sequence in geometry proposed for Grade 4 - 6 students. This analysis is mainly based on the Geometric Working Spaces model used as a tool to clarify and structure a coherent teaching of geometry.

Key-words: Geometry, Geometric Working Space, Pedagogical progression.

INTRODUCTION AND OBJECTIVES

As a teacher trainer, one of the authors (Assia Nechache) is in charge of several training sessions, especially in geometry, for in-service primary school teachers. Various activities are used during these training sessions and are selected from available resources written in French and dedicated to teacher training, such as the book *Concertum* (2002) edited by the French Commission for primary school teachers' training or journals like *Grand N*. The trainees expressed their interest and pleasure in discovering and doing these activities, but at the same time they were very anxious about devising and organising an overall progression of the teaching of geometry using these new, specific activities in their classrooms.

Indeed, it is usually possible to find ideas and examples of activities to implement in a classroom situation: resources such as those we mentioned above are relatively numerous. By contrast, very few are concerned with providing tools to help teachers reflect on and develop their own overall progression in geometry.

In these conditions, we may wonder how a teacher trainer can help teachers integrate new and interesting tasks in their classroom progression. This is by no means an easy question to answer, even for teacher trainers. We therefore focused our study on possible tools to help teacher trainers deal with this specific issue.

In another context, that of the research in geometry, the model of the Geometric Working Spaces (GWS) is used to understand and structure the various ways of thinking about the teaching of geometry throughout compulsory schooling. We wondered whether this model originally designed for education research could also be a tool for the training of primary school teachers, as it could help them conceive and implement a progression in geometry. Thus, as a first step, we decided to explore the use of the GWS model with teacher trainers to highlight the main elements that organise a long teaching sequence on a specific topic and see how it can help structure an overall progression.

In this paper, we first present the model of the Geometric Working Spaces and then report an analysis of a complex and long sequence of activities on the concept of circle. The analysis was conducted with a group of 27 teacher trainers during a working group devoted to the teaching of geometry. The analysis is mainly based on the GWS model and it allows us to highlight some key points to think about the teaching of geometry in elementary school. The reported experiment is not a product of pure mathematics education research according to the standards of didactic engineering, but can be considered as an action research process, and thus as an initial attempt to adapt and transpose a theoretical tool from research to teacher training education.

THE MODEL OF THE GEOMETRIC WORKING SPACES

The Geometric Working Spaces (GWS) and geometrical paradigms have already been presented, including during former CERME sessions. For a general presentation of paradigms, we refer to Kuzniak and Houdement (2003) where the three paradigms GI, GII and GIII are identified. An introduction to GWS was presented in a plenary lecture during the last CERME session in Turkey (Kuzniak, 2013). The GWS are now part of the more general framework of the Mathematical Working Spaces (MWS) described by Kuzniak and Richard (2014)[1]. We will summarise here some key points which we think are useful to understand the analysis that we will report in the paper.

The Geometric Working Spaces

The description of the geometric work done by students in school is the main purpose of the Geometric Working Spaces (GWS). As its name suggests, the geometric work is at the centre of the model and motivates the reflection on the teaching and learning of geometry. In this approach, the crucial function of educational institutions and teachers is to develop a rich environment which will enable students to solve geometric problems in an appropriate way.

To describe the specific activity of students solving problems in geometry, the GWS is organised into two planes or levels. The first, "epistemological" plane defines *a priori* expectations about the activity according to the requirements of the mathematical domain, in this instance geometry. As regards geometry, three interacting components are characteristic of geometric activity in its purely mathematical dimension:

− A real and local space as material support, with one set of concrete and tangible objects such as figures or drawings;

− A set of artefacts such as drawing instruments or software;

− A theoretical reference system based on definitions and properties.

The geometry that is taught and learnt at school is not a disembodied set of properties and objects reduced to signifiers which can be manipulated by formal systems – it is first and foremost a human activity. Therefore, it is essential to understand how communities of individuals, but also specific individuals, use and internalise their knowledge of geometry in their practice of the discipline. This implies a second, "cognitive" level centred on the subject viewed as a cognitive subject solving problems. Considering geometric activity, these processes are as follows:

− A process of visualisation related to the representation of both space and the material support;

− A process of construction and function of the used instruments (e.g. rulers, compass) and the respective geometrical configurations;

− A discursive process producing arguments and proofs.

This set of relationships could be described proceeding from the elements of the following diagram which, in addition, shows the relationships between the two levels with three different dimensions or geneses: semiotic, instrumental, and discursive.

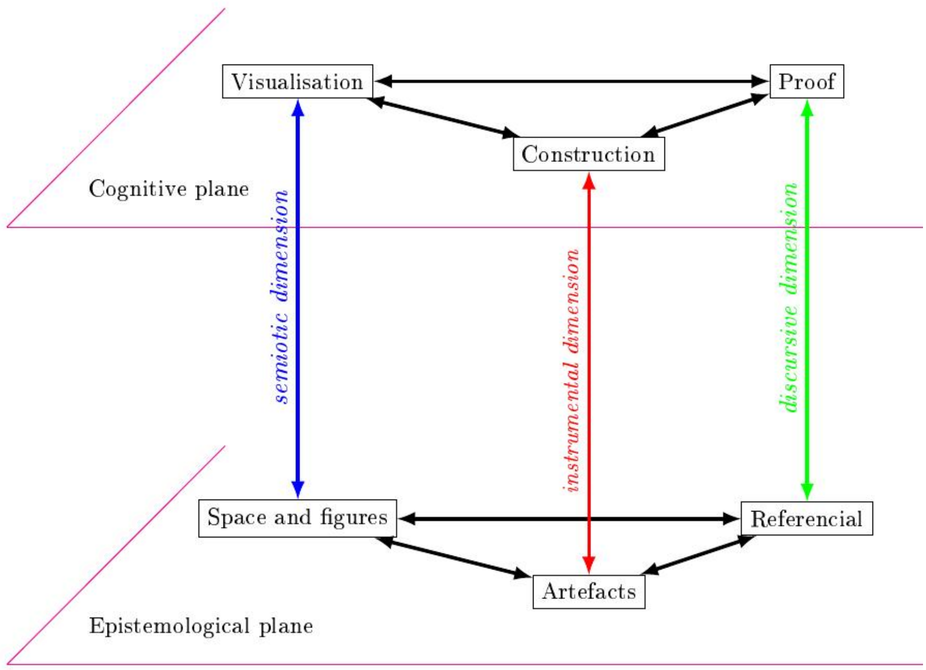


Figure 1. General diagram of the Geometric Working Spaces

Various types of input in the geometric work

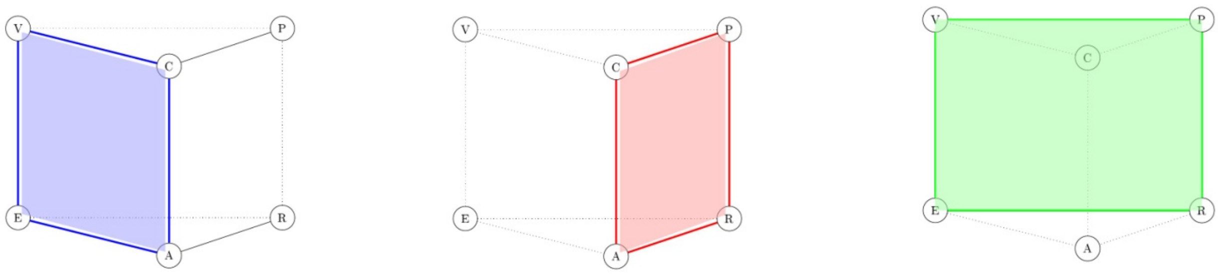
The above diagram shows three specific work dimensions between epistemological and cognitive planes which will require three specific genetic developments named geneses.

1. A figural and semiotic genesis that gives the tangible objects their status of operating mathematical objects;

2. An instrumental genesis that transforms artefacts into tools within the construction process, which is crucial in the case of geometry;

3. A discursive genesis of proof that gives a meaning to the properties used within mathematical reasoning.

The diagram in Figure 1 shows three vertical planes that match the connections between these dimensions and that will help us later to specify the precise geometric work existing in the GWS when students solve tasks given by their teachers. The three levels can be identified by the genesis they implement [Sem-Ins] (blue), [Ins-Dis] (red) and [Sem-Dis] (green). The objective of the analysis we present in this present was to understand precisely the nature and dynamics of these planes during the resolution of a series of geometric problems.



[Sem-Ins] [Ins-Dis] [Sem-Dis]

Figure 2. GWS vertical planes

DIDACTIC ANALYSIS OF A TEACHING SEQUENCE THROUGH THE GWS VIEWPOINT

According to the theoretical framework, we assume that a geometric work can be considered "complete" when a geometric entity is built throughout the three semiotic, instrumental, and discursive dimensions of GWS. For this reason, a geometric entity, here the "circle", will be considered a triplet [sign, artefact (material or symbolic), property]. As a result, building a progression in geometry first requires the identification of key geometric entities in the curriculum and then the analysis of the work associated with these entities in terms of GWS dimensions. We focus here only on the second point related to a long teaching sequence on the circle entity.

The selected sequence « Le cercle sans tourner en rond » was conceived and implemented by two well-known and experienced French scholars in the domain, Fénichel & Taveau (2008). It has been tested in various classrooms under different conditions and is relatively known by teacher trainers. Dedicated to Grade 4 – 6 students, the whole program is very ambitious and includes eight sessions – from half an hour to one hour – which can be administered over three months according to a spiral program. The main objectives of the sequence are to introduce the global notion of circle as the set of all points equidistant from a given point, the centre; to use this property to solve distance problems; to relate it to construction with compass used also to transfer distances. According to our analysis, the "circle" entity targeted by the sequence can be described with the triplet [circle as drawing, compass, equal distance].

Some activities of the sequence have been chosen for a common analysis with the 27 teacher educators in a working group setting. We here only report on the activities corresponding to classroom sessions 1, 2, 3, 4 and 7.[2] Each of the five selected sessions has been analysed using the GWS model. In particular, participants in the working group were asked to identify, if possible, the different input dimensions (semiotic, instrumental, discursive) in the geometric work and the favoured planes (plane [Sem-Ins], plane [Sem-Dis] and/or plane [Ins-Dis]). This identification highlights the dynamics of the geometric work during the different studied sessions and allows for the characterization of the overall implemented GWS.

We first detail the analysis of session 1.

The objective of the session is to highlight the fact that the circle is the set of all points equidistant from a given point, the centre. A varied amount of material and artefacts is made available: blank and tracing paper, twine, square set, compass...

Students are asked to draw 15 points at a given distance from a point A. During this phase of action, they need to place a point A on the white sheet and then a point B. After that, they need to place 15 points "situated at a distance from A which is the same as the distance of B from A". The geometric work starts in the plane [Sem-Ins].

Then, during a phase of formulation, some students' productions are displayed on the blackboard and discussed. The strategies they used to carry out the task are clarified and formulated. The objective is first to validate the notion of circle as discussed by the pupils based on the pupils' constructions. The notion of equidistance to a given point is expected to emerge. Some geometric terms are institutionalized and the characteristic property of the circle is given by the teacher and enriches the theoretical referential in the GWS.

The analysis made by the teacher educators is summarised in the following table.

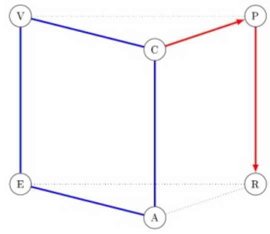
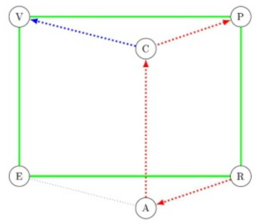
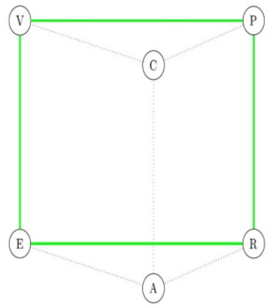
Sessions	Input in geometric work	GWS Diagrams
Session 1 The circle defined as the set of all points equidistant from a given point, the centre	The geometric work starts in the plane [Sem-Ins] and is concluded by the enunciation of the characteristic property of the circle which enriches the theoretical referential in the GWS. Properties and definitions of various figures used in this Grade are included in the referential. At this level, figures are generally introduced by "ostension"; it is worth noticing that such is not the case here : the proposed session clearly contrasts with traditional classroom sessions.	 [Sem-Ins] → Dis

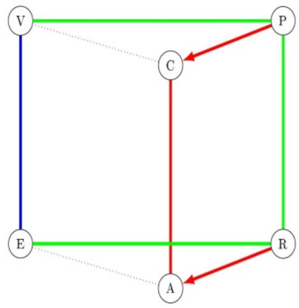
Table 1. The analysis of session 1

We have carried out the same work with other sessions. The different forms of the geometric work identified in the five sessions are presented in the following table and linked to a GWS diagram.

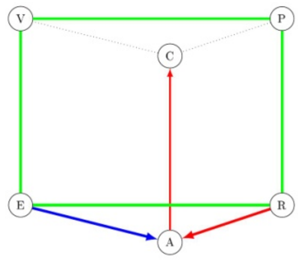
Sessions	Input in geometric work	GWS Diagrams
Session 2 Geometric problem solving with use of the stated property during session 1	The new theoretical property needs to be used to solve a problem. The input into the work is first rather semiotic but the use of a discursive proof is required to validate the solution. The use of drawing instruments strongly depends on the previous identification of the characteristic property of the circle. Artefacts are proposed, in option, as verification tools after the task is solved in the plane [Sem-Dis].	 [Sem-Dis] (→ Ins)
Session 3 Giving sense to Session 1 property by using it on freehand drawings.	The geometric work is this time in the plane [Sem-Dis] but with a clear discursive input because freehand drawings are used and can be considered as symbolic signs. Drawing instruments are set aside. As a consequence, the discursive dimension is now essential to the validation of the solution. The idea is to show that the "circle" is a theoretical object grounded in a characteristic property and not only an empirical object perceptively and	 [Sem-Dis]

<table>
<tr><td></td><td>instrumentally linked to a drawing. Validation is based on a discursive proof within Geometry I and appropriate to the primary school level, but it is paving the way for Geometry II, which is the challenge of geometry teaching in French secondary education.

As was the case previously, the authors still give students the opportunity to return to the experimental validation if it is necessary for their understanding.</td><td></td></tr>
<tr><td>Session 4

Discovering circle and disk uses to solve equal distances problems.</td><td>This time, the activity supposes a construction after modelling the problem. Once the task is interpreted, the geometric work is mainly located in the plane [Dis-Ins] using the theoretical referential: the characteristic property of the circle appears as a theoretical tool to build the solution. The data are provided in the semiotic register and the circle property ensures the validity of the solution.</td><td>

[Sem-Dis]→[Dis-Ins]</td></tr>
<tr><td>Session 7

Triangle construction using the circle property and introduction of the compass as length-transferring tool</td><td>The compass will acquire a new function. Initially considered as a tool dedicated to the construction of circles, the compass is used to transfer lengths and construct other geometric figures such as triangles. An enlargement of the use of the artefact is intended and it is related to the theoretical referential and the “circle” figure. The geometric work starts in the plane [Sem-Dis] and then enriches the instrumental dimension.</td><td>

[Sem-Dis] → Ins</td></tr>
</table>

Table 2. The analysis of the other sessions

To summarise, the geometric work is centred on the development of the notion of “circle” viewed as the set of all points equidistant from a given point and it is closely linked to the use of the compass. The epistemological plane can be defined by the triplet [circle as drawing, compass, and equal distance].

The geometric work conceived by the designers of the sequence is based, firstly, on the material artefact to bring out a property and enrich the set of theoretical tools. Then, the material artefact is set aside to promote a discursive reasoning using the theoretical notion of circle associated with the notion of equal distance.

At the end of the sequence, a return to the material artefact is operated to introduce a new use of the compass, which triggers a new circulation of the geometric work trough the different vertical planes. All the aspects of the work pertain to Geometry I but the sequence clearly paves the way for a prospective work in Geometry II at secondary school.

Teacher trainer viewpoints.

As mentioned above, the global analysis of the sequence has first been conducted by the two authors and then some of the activities have been chosen for a common work with teacher trainers during a working group: that ensures a stronger relevance to the final analysis which is based on the convergence of the different contributions made by participants.

In relation to in-service training and our initial question, the identification of the types of input in geometry seems successful in helping teacher trainers consider an overall progression in the teaching of geometry based on the organisation of a set of geometric tasks to promote a complete geometric work along the three GWS dimensions. Teacher trainers agree with this idea and underline that it is possible to have a global vision of geometry thanks to the GWS model. Moreover, teacher trainers have been aware of the importance of linking the different dimensions (semiotic, instrumental and discursive) within the geometric work.

Teacher trainers insist on the fact that an analysis using the GWS model is relevant to highlight the dynamic evolution of the mathematical work during the different sessions The evolution of the work is illustrated in the following global diagram.

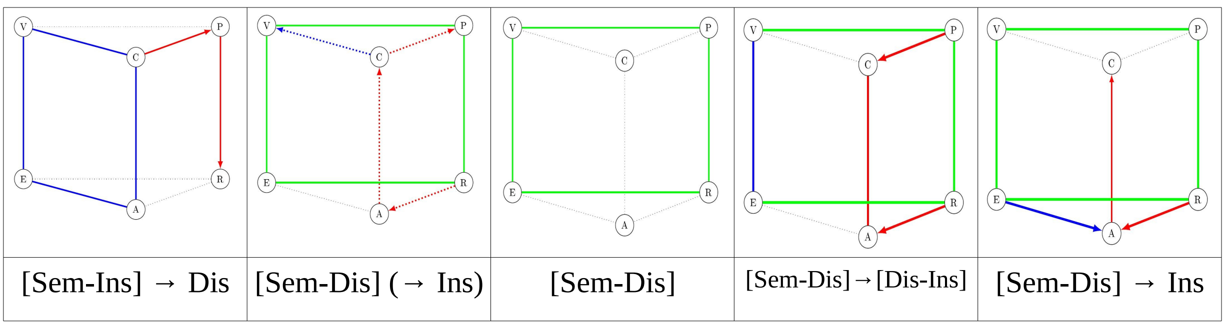


Figure 3. The dynamic evolution of the mathematical work during the different session

CONCLUSION

Our initial question was how to devise an overall progression in the teaching of geometry at primary school. As the issue is very complex even for teacher trainers, we started our study by working with them during a workshop. To deal with this issue, we introduced a twofold approach: first, the foremost mathematical entities targeted by the curriculum need to be identified; then, activities and tasks related to a specific entity can be analysed according GWS model to ensure a global consistency. In the paper, we focused on the analysis of a sequence of activities on the "circle" entity to identify the different favoured GWS dimensions. The GWS which appear in the selected sequence are structured around a set of tasks related to the triplet [circle-drawing, compass, equal distance]. This set of tasks mobilises different articulations[3] between the three vertical planes of the GWS diagram and gives birth to a real and complete dynamic cycle in the geometric work.

According to the teacher trainers involved in the study, this approach gives a global vision of the geometric work and highlights the choices of the designers. It allows teachers to assess the consistency of a sequence and permits them to discuss the choices made. For instance, some teacher trainers did not agree with the idea of focusing on the characteristic property of the circle at this level of schooling. We may then wonder what are the consequences of other inputs giving priority to the use of software or spatial activity on the geometric work. With the GWS tool, it seems possible to discuss the « best dynamics » to favour the geometric work among teachers and pupils.

The question remains of the use of the model in teacher training and teachers' practice. We hypothesize that it is more adapted to teacher trainers than to teachers. Indeed, the priority in teacher education is to explain the mathematical content involved in teaching sequences but our reflection on the didactic transposition of the GWS theoretical model should be furthered and is one of our prospective research topics. What is clear from our experience is that it is not necessarily required to present the GWS model in depth to teacher trainers: a short introduction based on the GWS diagram has enabled us to conduct our analysis with convincing results.

NOTES

1. For readers interested in the development of the model see the web-page http://turing.scedu.umontreal.ca/etm/documents/Actes-ETM3.pdf where the proceedings of ETM3 can be found.

2. The whole sequence is online on Alain Kuzniak's web-page: http://www.irem.univ-paris-diderot.fr/~kuzniak/publi/Publications. Even though the text is in French, we hope that the reader can understand the main phases of the different geometrical activities.

3. These smooth and graduate transitions between planes or dimensions are called "fibrations" in the GWS model.

REFERENCES

COPIRELEM (2002). Concertum, 10 ans de formation des enseignants du premier degré. *Paris: Ed Arpeme.*

Fénichel, M. & Taveau, C. (2009). *Enseigner les mathématiques au cycle 3. Le cercle sans tourner en rond.* DVD, CRDP Créteil.

Houdement, C. & Kuzniak, A. (2003). Elementary geometry split into different paradigms. *Proceedings of CERME. Bellaria, Italy.*

Kuzniak, A. (2013). Teaching and Learning Geometry and Beyond…*Plenary conference at Cerme8.*

Kuzniak, A. (2014). Understanding Geometric Work through its Development and its Transformations. *In Transformation – A Fundamental Idea of Mathematics Education, Rezat S. Hattermann M., Peter-Koop A. (eds).* pp. 311-326. New York : Springer.

Kuzniak. A & Richard, P.R. (2014). Spaces for Mathematical Work. Viewpoints and perspectives. *Revista latinoamericana de investigación en matemática educativa, vol 17.4.*