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SPACES FOR GEOMETRIC WORK: FIGURAL, INSTRUMENTAL, AND DISCURSIVE GENESES OF REASONING IN A TECHNOLOGICAL ENVIRONMENT

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ABSTRACT. The main goal of this research was to assess the effect of a dynamic environment on relationships between the three geneses (figural, instrumental, and discursive) of Spaces for Geometric Work. More specifically, it was to determine whether the interactive geometry program GeoGebra could play a specific role in the geometric work of future teachers. The training reveals education students' use of intuitive and deductive reasoning as well as their degree of cognitive flexibility in using different facets of geometrical work. The study shows that a perspective on the development of geometric reasoning must be integrated in the development of skills for future teachers. In particular, a focus on the construction of a discursive genesis linked with elements of figural genesis must be added to the traditional focus on instrumental genesis. This reasoning requires in particular a reflection on the role of properties in developing a form of geometry directed by practical (Geometry I) or more axiomatic (Geometry II) goals.

KEY WORDS: geometry, initial teacher training, mathematics, professional knowledge, Space for Geometric Work, technological environment

Introduction

The main goal of this research was to characterize and define the exact nature of the geometric work done by future teachers in technological contexts such as interactive geometry programs (GeoGebra). Using the theoretical framework of Spaces for Geometric Work (SGW), an initial research project with a group of student teachers found their geometric work to be incomplete. Within this framework, a geometric work is considered complete when students are able to master the skills involved in three geneses—figural, instrumental, and discursive—with flexibility and adaptability. The role of the instrumental genesis can be problematic among students who rely on the figural and discursive aspects. This is especially true when there is no congruence between the theoretical tool and the computer tool. The incompleteness of student teachers' geometric work, as identified by the first study, has been tested and verified with a complementary and larger study involving four groups. The entire study, along with its theoretical background, will be presented in this paper.

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Theoretical Considerations

The framework of geometrical paradigms and Spaces for Geometric Work (Houdement & Kuzniak, 1999a; Kuzniak, 2006, 2011; Kuzniak & Rauscher, 2011) describes the work that people (students, teachers, mathematicians, etc.) perform when they solve geometric tasks. To formulate and communicate their solutions, people need to share a common set of beliefs, techniques, practices, etc. This shared set of beliefs is called a paradigm (Kuhn, 1966). To study the geometry taught at school, three geometrical paradigms, labeled Geometry I, II, and III, have been considered. The numbers do not constitute a ranking: one is not preferable to another. The perspectives of these paradigms are different, and an individual's choice of path towards a solution is determined by the purpose of the problem and the investigator's viewpoint.

Geometry I: Natural Geometry. This geometry has the real and sensible world as a source of validation. In this geometry, an assertion is supported using arguments based upon experiment and induction. Proofs could rely on drawings or on observations made with tools such as rulers, compasses, and protractors. The development of this geometry was historically motivated by practical problems.

Geometry II: Natural Axiomatic Geometry. Geometry II, whose archetype is classic Euclidean Geometry, is built on a model that approaches reality. Once the axioms are established, only proofs developed within the system of axioms are valid. The system of axioms could be incomplete: The axiomatic process is a work in progress with modeling as its perspective.

Both of these geometries are closely linked to the real world, but in different ways. This differs from Geometry III (Formal Axiomatic Geometry), where the system of axioms itself, disconnected from reality, is central. The system of axioms in Geometry III is more concerned with logical problems and tends to contain "intuitive" axioms without any reference to perceptive evidence such as convexity or betweenness. This last Geometry is now rare in compulsory schooling, but remains the implicit reference frame of teachers who have studied mathematics at the university level, where this formal and logical approach is common.

When specialists are trying to solve geometric problems, they switch repeatedly between paradigms. They use figures and tools in various ways, including as a source of knowledge or of validation of some properties. However, specialists always know the exact status of their hypotheses and the level of confidence they can give to each of their conclusions. By contrast, when students complete the same task, we cannot be sure about their ability to use knowledge and techniques related to geometry. We must observe geometric practices within a classroom to understand common uses of mathematics tools.

In order to describe the complexity of these practices, the notion of Space for Geometric Work (SGW) was introduced. The SGW structures the work of people solving geometry problems. It establishes the reference to the complex setting in which the problem solver acts. Individuals can be experts (the mathematicians), students, or senior students in mathematics. The makeup of a SGW will vary with the education system (the intended or reference SGW), the school circumstances (the implemented or appropriate SGW), and the practitioners (students' and teachers' personal SGW).

Two interconnected planes have been introduced to structure the SGW: the epistemological plane and the cognitive plane. In the epistemological plane, three elements intersect:

A real and local space as material support, with a set of concrete and tangible objects
A set of artifacts (1), such as drawing instruments or software
A theoretical frame of reference based on definitions and properties

These three components are organized and interconnected in light of a precise goal that depends on specific geometrical paradigms. With these paradigms, an epistemological viewpoint on Geometry is introduced as a way to organize knowledge and present the rational knowledge under codified standards. This knowledge in return contributes to the construction of the objects' meaning in a determined direction. This justifies the name of "epistemological plane" for this first level (Fig. 1).

The cognitive plane was introduced to describe the cognitive activity of a single user. The idea of three cognitive processes involved in geometrical activity is adapted from Duval (2005).

- A visualization process connected to the representation of space and material support
- A construction process determined by instruments (ruler, compass, etc.) and geometrical configurations
- A discursive process which conveys argumentation and proofs

Both planes, cognitive and epistemological, need to be articulated in order to ensure a coherent and complete geometric work. This process

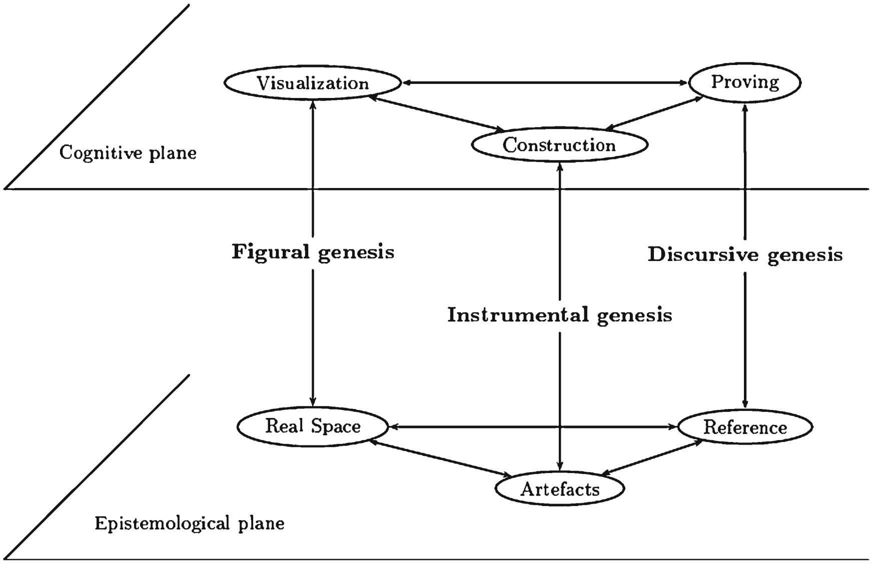


Figure 1. Diagram of geometric work supported by three geneses

assumes the presence of some transformations that are possible to define through three fundamental geneses strictly related to our diagram.

A *figural genesis* is the semiotic process associated with visual thinking and transition from a syntactic to a semantic perspective on geometric objects. It informs geometry where figures are the visual supports favored by geometrical work. It provides the tangible objects their status of operating geometrical objects. Duval (2005) mentioned two levels of operations relevant to this process. The first one is iconic and is associated with the perceptual recognition of shapes, while the second one is related to a more abstract interpretation of signs with the idea of figure as a symbolic object.

An *instrumental genesis* transforms artifacts into tools within the construction process. Computers have completely revolutionized the question of the role of instruments in mathematics by facilitating their use and offering the possibility of dynamic proofs. Based on Rabardel's work (1995) on ergonomics, Artigue (2002) stressed the necessity of an instrumental genesis with two main directions that can be inserted into our frame. The upward transition, from the artifacts to the construction of geometric configurations, is called *instrumentation* and describes users' manipulation and mastery of the drawing tools. The downward process, from the configuration to the adequate choice and the correct use of one instrument, is related to geometric construction procedures and is called instrumentalization. In this second process, geometric knowledge is engaged and developed.

In *discursive genesis of proof*, properties used within mathematical reasoning are afforded meaning. The purpose of this genesis is to engage a bidirectional validation process: a deductive, proving discourse supported on properties structured into the reference frame or, conversely, the identification of properties and definitions to be included into the reference frame after instrumental or visual treatments.

Different geneses do not operate separately but need to interact in order to give the geometric work a meaning that the present study aims to highlight. Using the diagram of the SGW, Coutat & Richard (2011) have brought up three vertical planes which appear in the representation and can help us to make more explicit the articulation existing between the geneses.

Plane (Fig–Dis) Associated to Figural and Discursive Geneses

A close connection between figural genesis and discursive genesis is crucial in developing a geometry that exceeds a simple iconic vision on objects. According to the priority given to the figural or discursive line, two kinds of approaches are possible. When the focus is on the figural side, mereologic and visual transformations structure the description of drawings and organize a perceptive reasoning. Conversely, if the focus is on a hypothetical and deductive proof based on properties, figures and visualization play a heuristic role.

Plane (Ins–Dis) Associated to Discursive and Instrumental Geneses

In this plane, the crucial point is the question of the proof being based on experiment or on pure deductive argumentation. If conclusions are drawn from data given by instruments, we shall speak of an experimental proof. In the other way, if the proof is based on a theoretical referential, instruments are used to illustrate or to construct geometric configurations.

Plane 3 (Fig–Ins) Associated to Figural and Instrumental Genesis

This plane has recently been given more importance due to the appearance of Dynamic Geometric Software. Dynamic tools increase the capacity of exploring configurations and discovering new geometric properties. The exploratory approach already existed but only with experienced students. In this case as well, two ways of working can be observed: the one is more oriented towards the construction of figures satisfying some conditions and the other towards the interpretation of data given by artifacts.

Several of the interactions identified in our study can be understood and interpreted with geometrical paradigms. The nature of geometrical objects, the use of a number of construction techniques, and the validation modes accepted are determined by the specific paradigm used by students. Conversely, using SGW diagrams allows to describe students' evolution and oscillation between paradigms when problems are solved. Indeed, it is well known that students show a great flexibility when they argue in Geometry, and it is possible to have information about this flexibility with successive SGW diagrams: when it occurs and what the teacher's role is in the evolution and the transition from one paradigm to the other.

Research Question and Other Investigations

Research into teaching in technological contexts (Laborde, 2001; Lagrange, 2009) shows that students have little or no knowledge of the teaching of mathematics; they are unaware of the development of mathematical notions in teaching situations, and they have difficulties in using software in a learning situation. This makes it necessary to integrate information on using software to teach into teaching training. Our study focuses on the role played by a technological environment on the development of geometrical competencies for student teachers.

Some studies on math learning in technological contexts have used geometric paradigms (Coutat, 2006; Mithalal, 2010) and geometrical work spaces (Coutat & Richard, 2011; Gomez-Chacon & Kuzniak, 2011) to interpret the actions of student reasoning with Dynamic Geometric System (DGS). In most studies, the instrumental approach (Artigue, 2002; Lagrange, 2009; Trouche, 2000) is preferred to studying a given task within a technological context. Both perspectives, that of SGW and of an instrumental approach, however, are complementary for understanding the development of geometric work and thinking. The instrumental approach allows us to address specific difficulties related to the use of technology, while the SGW is more sensitive to the cognitive and epistemological construction of a specific work in geometry.

More precisely, the structure and the relationships of these figural, instrumental, and discursive types of genesis within students' personal Space for Geometric Work will be identified for the case of training that requires the use of dynamic geometry. The influence of GeoGebra use on the student will be examined in order to study the interplay between Geometry I and Geometry II, as well as the use of properties and devices.

From the perspective of training students in the construction of the strategic knowledge of a teacher using technology, the difficulties encountered in this training and which weaken the personal Spaces for Geometric Work will be explained. The final goal is to understand more their personal SGW to develop a good "appropriate" SGW in the classroom.

Presentation of Task and the Research Methodology Used

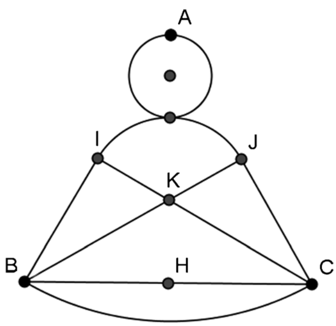
Study Group and Presentation of the Task

The population consisted of 98 student teachers (65 women and 33 men) in four training groups established by the academic institution. The task was initially assigned to 30 students (group A), and the first data analysis was based on their solutions. Of the 98 students, 81 were able to solve the "Bell" problem, while 17 were not.

The "Bell" was proposed as a didactic situation to the students. It was adapted from Houdement and Kuzniak (1999b). As will be shown, this problem highlights the structure of and connection between Geometry I and Geometry II. Initially, the problem was proposed in a paper and pencil environment. We adapted it to a technological environment with the use of GeoGebra software.

The statement of the task was the following:

Enlarge the following bell in such a way that A′H′ measures twice AB. Draft a procedure for the resolution of the problem given. Some clues which might help you to draw the bell are the following:



1. Observe that A is on the straight line BI and on the straight line CJ.
2. H is the halfway point of BC.
3. The angles IBC and JCB measure 60°.
4. The angles BIC and CJB are right angles.
5. BC belongs to a circumference of a circle whose center is A.

The bell was drawn on paper provided to students. The students were asked to construct the original bell on the screen before attempting to make a larger version. They were free to use paper and pencil. The students next were asked to build the enlarged drawing on the same screen as the initial one. From then on, they were encouraged to see the dependence between the drawings in the context of the software. Seeing this changes the nature of the task, the type of SGW, and the mathematical and technological level. The "Bell" training activity was carried out over two sessions. In the first session, students were given the problem and asked to describe their approaches to resolving the problem on protocols including: steps in the resolution, explanations of the difficulties they might face, and strategies they would use in order to solve with paper and pencil and with the computer. They were also asked to describe their emotions and difficulties while solving the problem and to record these in the written solution.

During the first session, the majority of students focused mainly on the construction of the initial bell. The second session was used for working on the enlarged bell and discussing common approaches to the process and the difficulties that arose. This session was recorded, which enabled us to observe students' activities, paying special attention to the interactions between the students and the teacher educator.

Methodology and Data Analysis

The first step in data analysis was to classify the construction methods of the 30 students in group A. Three classes of answers were identified among these students to construct the initial Bell and four methods to enlarge it. We then progressed to a larger analysis of all 98 students. This second analysis was based on the initial classification and added a set of variables related to properties and difficulties in the geometric work: using properties; typologies of the solutions for the initial and for the enlarged Bell; learning difficulties in the task, emotions, and difficulties with the artifact; and difficulties in the global control of the different geneses of geometric work.

The second analysis used a qualitative approach based on a cross examination of solutions by the authors and statistical data mining analysis.

(a) For each type of procedure, the two researchers conducted a separate analysis using SGW diagrams to determining the work flow to performing the task. The idea was to identify geneses (instrumental, figural, and discursive) involved in the process and draw the corresponding diagram by identifying the existing articulations. The direction of an arrow was only fixed where a simple explanation was possible. The following hints were used as guides in the analysis of each genesis.

For instrumental genesis. To identify the use and role of instruments and techniques.

Up. Priority is given to drawing tools, and the construction method is supported by exploration with artifacts.
Down. Appropriate instruments are chosen by students after a global a priori analysis of the construction.

For figural genesis. To identify the use and role of figures.

Up. Giving a mathematical meaning to a drawing and perceptively identifying properties and definitions (from syntax to semantics).
Down. Visual work aiming at deconstructing the global and abstract notion of figure into simple elements and signs (from semantics to syntax).

For discursive genesis. To identify validation methods and use of discourse.

Up. A discourse deductively organized and based on the theoretic referential.
Down. Enrichment of the theoretic referential after visual or experimental approach.

(b) The researchers' findings were compared and discusses where disagreement was detected. Joint work focused on identifying possible links between construction procedures for the initial and the enlarged bell.

The data mining was done with the statistical SAS 9.2 package for extracting nontrivial and implicit information from data.

Based on the first analyses and complemented by the second, the following conjectures and hypothesis were developed, explored, and tested:

Hypothesis 1: Students whose geometric work is supported first by discursive geneses tend to work in Plane (Dis–Fig) and face difficulties and constraints when transiting through the instrumental genesis to finish the task.

Hypothesis 2: Students whose geometric work is supported first by instrumental genesis used the software and refer to figural genesis to discover and identify properties, and that can contribute to cut off the links to discursive genesis.

Typologies of Constructions

First ("Typologies of Students' Solutions for the Construction of the Initial Bell" section), the typologies of constructions made by the students to draw the initial bell will be presented. Then ("Typology of Students' Solutions of Enlarging the Initial Bell" section), the enlargement procedures obtained will be listed. This section will be concluded with some results ("Results Combining Both Procedures" section) combining both procedures.

Typologies of Students' Solutions for the Construction of the Initial Bell

Three different types of solutions were obtained: solutions with "rulers and compasses," solutions with "angles," and solutions with a "regular polygon." These procedures make it possible to trace the equilateral triangle underlying the construction. All the procedures followed to complete the bell are very similar.

"Ruler and Compass" Solutions (RC). These types of solutions are the transposition of the ruler and compass construction in the paper and pencil context to the software. These are Euclidean-type constructions in which the software constructs the circles and adds the components and the relationships between the points of intersection. These decisions are guided by spatial considerations. Below are the details of the construction of the triangle as found in the productions of the students:

1. Create points B, C, and the segment joining them.
2. Draw the circle with radius BC with B at the center, and the circle with radius BC with C at the center. Take the "higher" intersection point of the intersections of the circles denoted by A. Then, ABC is an equilateral triangle.
3. Find the points I, J and draw perpendicular lines at sides AB, AC of the triangle which pass through the vertices C and B, respectively.

Angle Solutions (AN). Clue 3 is used in these solutions in order to indicate the measurement of the angles. The construction can be described as follows:

1. Create points B, C and the segment joining them.
2. Use the GeoGebra angle command "Angle given its measure" first marking point B, then point C, and then indicating the desired angle

measure of 60°. The program then creates a point which is redesignated A. Then, ABC is an equilateral triangle.

3. Find points I, J. The majority of students who use this construction use the same method as in type 1 solutions. Others use the concept and command “mediatriz” (“perpendicular bisector”).

One variant used by some students was to use the command for rotation around a point with a given angle. This variant can probably be explained by the similarity in the GeoGebra commands “rotation (object center)” and “angle with a given measure.” The rotation command constructs a point A such that the angle (BCA) is equal to 60° and CA=CB. One side of the angle is not drawn on the screen. In effect, this command constructs the image of B under a 60° rotation around C.

“Regular Polygon” Solutions (PR). This last procedure is only possible within a digital environment, as it uses the command “Regular Polygon” to draw an equilateral triangle in a single operation.

1. Create points B, C and the segment joining them.
2. Use the command “Regular Polygon,” indicating one side of the polygon and the number of sides.
3. Find points I, J, using the tool which finds the midpoints of each of the sides of the triangle.

Typology of Students’ Solutions of Enlarging the Initial Bell

Four types of solutions appear: the “Thales” solution (TH), the Pythagoras solution (PI), the Angles solution (AAn), and the Homothety solution (H).

The Thales Solution (TH). The students in this category use Thales’ theorem to obtain the distance and then the same method to construct the enlarged bell as they had used to draw the initial bell. The reasoning relies on the classical figure which makes it possible to see Thales’ theorem in the case of the triangle. This makes it possible to obtain the following general formula:

$$B'C' = 4\ AB \times BH/AH.$$

We should point out that some students use paper and pencil, and others use direct reasoning with the GeoGebra software.

The Pythagoras Solution (PI). This involves applying the Pythagorean Theorem in order to find the distance. The rest of the construction follows the same rules as those preceding. In this case, reasoning is internal to the triangle A′B′C′ without modification to the figure. The result obtained connects B′C′ to AB.

$$\text{As}\quad A'H' = 2AB \quad \text{then}\ B'C' = \frac{4}{\sqrt{3}}AB$$

In this Pythagorean Theorem-based solution, the students must perform algebraic calculations. Some students use paper and pencil, and others reason directly on the computer. Applying Thales' theorem makes it possible to obtain a construction without knowing the values, but using the Pythagorean Theorem makes such calculations necessary.

The Angles Solution (An). This construction is based on the height of the triangle, which must be equal to 2AB. We now show the more general solution used by the students:

1. Draw the height AH′ with distance 2AB by using the command "segment given its length and first endpoint."
2. Find two straight lines from A which form an angle of 30° with the height, using the command "angle given its measure."
3. From H′, draw a perpendicular to the height. The points of intersection with the straight lines constructed in the previous step are B′ and C′, giving the final two vertices of the triangle.

The Homothety Solution (H). Finally, one strategy which might appear to be evident is drawing a homothety, or dilation outward from a point, based on the condition given in the problem: AH′=2 AB.

The construction is specified below:

1. Add a point to the exterior of the bell. This will be the homothety point.
2. Use the command "dilate object from the point indicated by a scale factor." This enlarges the entire bell.

The students worked with the 2.0 version of GeoGebra and had trouble finding the command in the GeoGebra menu. In its Spanish version, there is no "homothety" command, but there is a "dilatación" command. Once this command is identified, it required nonevident implementation in order to handle the software properly and resolve the construction of the problem.

TABLE 1

Procedures used

	Thales	*Pythagoras*	*Angles*	*Homothety*	*Unsolved*	*Total*
Ruler and compass	5	6	7	3	4	25
Regular polygon	4	–	9	2	3	18
Angle	7	–	44	1	3	55
Total	16	6	60	6	10	98

Results Combining Both Procedures

A quantitative summary of the procedures used to construct initial and enlarged bells is given in Table 1. The relationship between the two tasks was used to characterize geneses interactions involved as a whole and subsequently in the stable forms of students' SGW. These characteristic SGW are described in the following section.

Some General Characteristics of the SGW

A relationship is established between the two tasks for determining the role played by geometric work in the genesis of the activity as a whole. Two main Spaces for Geometric Work are identified on the grounds of data. A first SGW is governed mainly by a discursive reasoning, and the second promotes an instrumental view on geometry activities. Each SGW is presented in items 6.1 and 6.2 and illustrated with a characteristic example, along with some variations and students' difficulties. In the section "Evolution and Oscillation Between Geneses," the alternations between geneses are studied in item 6.3 based on the special case of homothety procedures, and a third SGW supplementing the second is exhibited.

A Geometric Work Supported by Discursive Genesis of Reasoning

Geometric work is based here on a general meaning of proof grounded on a set of definitions and properties. This SGW is directed by discursive genesis and relates to the figural genesis used to recognize subfigures and identify properties in configurations. The actual precise construction appears as a result of simply performing this work. Geometry II concerns this kind of SGW. Students face difficulties when they do

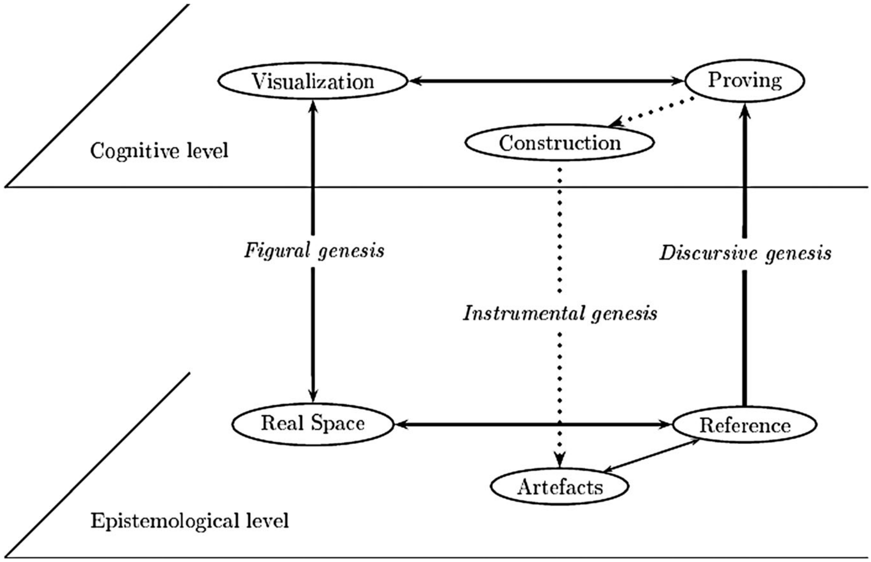


Figure 2. Geometric work supported by discursive genesis

not master DGS or when instruments do not fit exactly to and are not congruent (in the sense of Duval, 2005) with the intended construction procedures (Fig. 2).

Students using a construction based on RC to draw the initial bell and then Pythagoras Theorem (PI) for the enlarged bell are representative of this SGW. Their reasoning is based on hints (1) and (3)

(1) A is on the straight line BI and on the straight line CJ.
(3) The angles IBC and JCB measure 60°.

From these observations and using definitions and properties, they deduce that ABC is an equilateral triangle. They then consider constructing A as the intersection of circles with center in B and C or alternatively on the perpendicular bisector of BC. Both constructions are based on the fact that A is at the same length of B and C. The work is guided by discursive genesis. The available dynamic geometric tools are not taken into account by students in solution of the problem. Students are supported on previous teaching experience and experimentation with ruler and compass using paper and pencil. It is difficult to characterize the construction techniques used by students from the standpoint of instrumental genesis. This is so familiar that it is in fact a naturalized technique. The preparatory sessions for using the software ensures that the students do not have any problems with implementing this procedure.

To better understand the real place of instrumental genesis in this first geometric work, a study of how students using RC have enlarged the bell is necessary. All the procedures have been used: Thales, Pythagoras, Angles, and Homothety. For us, the Pythagoras procedure is representative of a complete geometric work integrating the three geneses according to this SGW. From visual analysis of the figure, students focus on the right triangle which is viewed from a theoretical viewpoint as the starting point of a deductive reasoning oriented to construction. The construction is based on a classical theorem, the Pythagorean theorem with a discursive genesis. However, to apply the theorem, students must carry out substantial work to visualize the figure, including observing heights and midpoints of the sides. They needed to perceive the figure not as a particular case, but as a figure which serves as support for the construction. The GeoGebra software does not allow an alternative route for this calculation.

Close to this approach are students using RC/Thales. As previously, the construction is based on a classical theorem, which means it is based on properties. Some of these students consider proportional objects and Thales' theorem first in a paper and pencil environment and not in GeoGebra. The paper and pencil context provides a heuristic basis (the drawing does not maintain the proportion), and geometric work is directed by the discursive genesis supported by the figural genesis. Among the students who use this procedure, some do so directly with GeoGebra. These students maintain the proportions of the image by using the commands "segment given length and endpoints" or "distance." In this case, the work remains oriented by Geometry I.

As stated above, some of the students solved the problem on separate page on paper. When entering the distances in GeoGebra, the students did not know how to handle the lengths of the segments of the sides. They needed improvement in this area, with specific work on implementation in order to enlarge the bell. The difficulties which appear in this type of solution involve mastering the device while passing to the classical geometric thought of Geometry II and will explain some blockage due to a lack of instrumental abilities.

Using the case of Celia, we illustrate how the mastering of the software influenced students who have greater interaction between figural and discursive genesis and have consequences on the global geometric work.

Like some students in the same group, Celia mobilized an explicit reasoning using Thales' theorem and obtained the correct value of B′C ′ on paper.

$$B'C' = \frac{4 \times BH \times AB}{AH}$$

In this case, she became blocked whenever she needed to transfer this result to GeoGebra, which does not allow a formal treatment of the outcome she wanted. She used Thales' theorem to obtain the distance and again the same method to construct the enlarged bell as the one she had used to draw the initial bell. Using Duval's formula, the solution with GeoGebra is not congruent to the paper and pencil process. This idea was well apparent in Celia's discourse about her process.

The construction of the first bell, based on the indications given in the setting of the problem did not entail any problems for me and I did all the work directly with Geogebra, without any reasoning given on paper.
The main problem arose in the construction of the second bell. The difficulty lay in how to apply the paper relationship to Geogebra.
It was not difficult to find the relationship side B'C' must comply with, but the problem arose when this idea had to be translated for use by the program. I knew the properties on paper, but it was difficult for me to transfer this to the computer. Finally, in order to solve this problem, I defined a magnitude with the B'C' measurement, a task which I found a little difficult to handle with GeoGebra.

Students like Celia attempted unsuccessfully to use the distance command to calculate the side B′C′ and finally produced a solution entirely grounded in the angles domain. In the development of their SGW, they were stopped by difficulties in using the tools.

In short and to complete our hypothesis 1, a significant part of students whose geometric work is supported first by figural and discursive geneses (Plane Fig–Dis) encounter difficulties and constraints when transiting through the instrumental genesis, especially when the use of the tools is not congruent to their solution. By contrast, students who do not face such difficulties develop a figural decomposition adequate to an instrumental decomposition and exhibit a complete geometric work.

A Geometric Work Promoting Instrumental Genesis

Student's geometric work is now guided by the construction tools given by the software. The figure is visually decomposed to make possible the use of drawing tools, and figural genesis on the figure is guided by students' instrumental abilities. Attempts to enlarge the bell are made

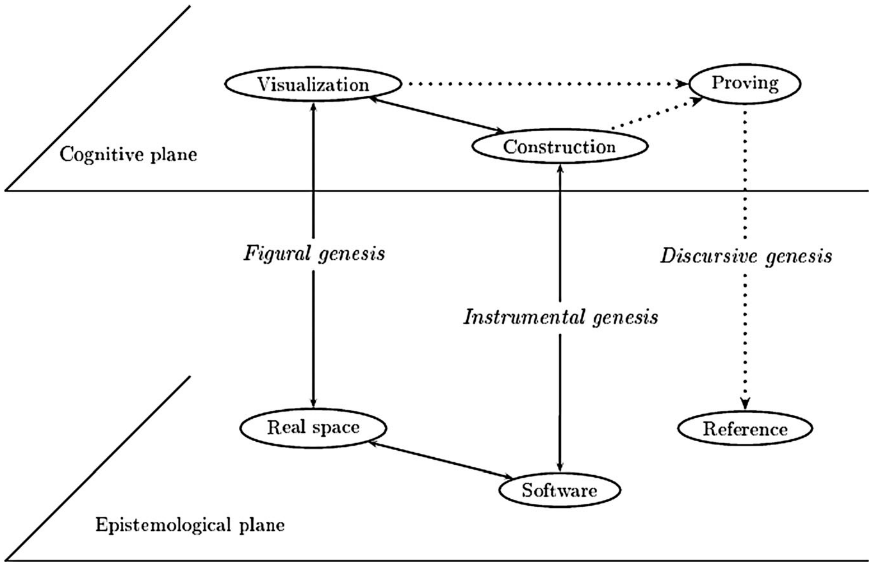


Figure 3. Geometric work promoting instrumental genesis

directly with tools without any try to calculate ratios and some blockage can be observed when it is necessary to base the work on discursive genesis to achieve the construction. The SGW is summarized in Fig. 3. The point is here to see how the plane (Fig–Ins) relate or not to discursive genesis.

The class of students using Angles/Angles is the most representative of this approach. This type of construction relies on the overall perception of the figure and on some properties of equal angles. This solution is equivalent to the use of the protractor with paper and pencil, but the construction proposed in GeoGebra is not congruent with the construction with paper and pencil, which requires greater skill with the software. Two predefined tools can be used in the software to construct the angle and the triangle: “angle with a given measure” or “rotate an object around a point, with a given angle.” These indicate two ways of working in geometry. The first approximates the way in which the protractor is used and refers to Geometry I work, and the second falls more within the domain of geometric transformations within the workspace of Geometry II. Most of the students have used the first procedure.

It should be pointed out that the invariant angle is useful for enlarging the figure, as it is precisely the angle which can give a direct construction from the segment A′H′ (segment with a length equal to the double of the length of AB). Based on this and on knowledge used to construct the initial bell, students were able to achieve the task but they showed a real lack of explicit discursive genesis. Using the command "angle with a given measure", the point A is constructed as the intersection of the lines making a 60° angle with BC with any reference to the equilateral triangle or identification of properties about bisector, height and median in such triangles.

To construct the bell, students use the symmetry of the equilateral triangle to construct the two straight lines that form an angle of 30° with the height. This involves using a property, but also performing visualization work on the figure. Another property which justifies the construction (but is still implicit) is that of the invariance of the angles. In this procedure, the students remain in a Space for Geometric Work oriented by Geometry I in order to construct the enlarged figure, guided by perception and instruments. By contrast and quoted above, some students used the command "rotate an object around a point, with a given angle" and have developed a discursive genesis to see the equality of segment lengths and the relationship with rotation.

We relate to this SGW, students using Regular Polygon even if some differences are observed in the initial construction of the bell and after. The solution is based on the possibility offered by the software to directly construct an equilateral triangle, viewed as a regular polygon with three sides. In order to find the "Regular Polygon" command, students must browse the GeoGebra menu and show their capacity to explore (do-it-yourself capability, or "bricoleur" in the sense of Trouche 2000). They focus on instrumental genesis but through the use of a specific and global tool after observing an equilateral triangle as a regular polygon with three equal sides. In addition, the same procedure cannot be adopted for the enlargement construction, for which students must change their strategy. We observed whether properties are used explicitly, and students remain within the instrumental domain, or whether they return to paper and pencil. From the study of their productions, it can be pointed out a general lack of discursive genesis. For instance, the point K is not constructed as the orthocenter or barycenter of an equilateral triangle except for few students. On the same way, they needed to articulate figural and instrumental genesis to construct arcs of circle centered on K and extremities I and J but they did not argue on the tangency at this point to ensure the right angles at these points.

As in the previous section, students faced instrumental difficulties with software commands mathematical meaning. A specific one appeared in this approach based on angles with understanding relationship between objects in dynamic geometry. It relates to what Yerushalmy and Varda (2004) call parent–child relationships and may have impacted the final

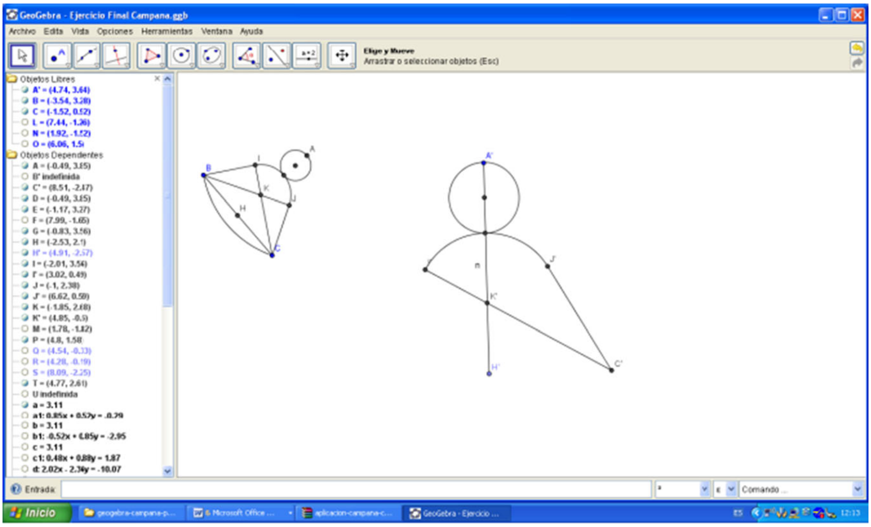


Figure 4. Priscila's solution (segment A′H′)

construction of some students who used a construction based on angles. These students encountered problems when they wanted to enlarge the figure, and the triangle suddenly disappeared. They had built an initial segment, followed by a height with length 2AB perpendicular to the middle of this segment, and finally the points B′ and C′ as the intersection of two straight segments that form an angle of 30° with the height. In this case, the points B′ and C′ depend on the initial segment. When the students enlarged the figure, the lines no longer cut the segment, and the points B′ and C′ disappeared. It was difficult for students to understand this difficulty (Fig. 4).

To conclude and complete our hypothesis 2, the geometric work promoting instrumental genesis seems to make difficult the use of properties even when it would be useful to understand some blockages or to give shortcuts to achieve the construction. Like squirrels on running wheel, students risk to stay blocked in the plane (Fig–Ins). Note that the difficulties were sometimes overcome directly by students as they explored or mastered the software. But in most cases, the teacher educator's intervention was necessary. Understanding all these blockages, as well as ways to overcome them, can also be used to support future teachers for teaching in secondary school. And their capacity to explore the software means that they remain in a Geometry I SGW where the two geneses linked to the device and global visualization are privileged. To conclude, it has been noticed that when students work on perceptual apprehension, using DGS, that is used to see figures and iconic visualization, they not always reach to operative apprehension, having consequences for discursive genesis, it can cut the links with the discursive genesis. Some difficulties with the software prevented the use of properties.

Evolution and Oscillation Between Geneses

Grounded on our data (Table 1) a special attention must be paid to procedures using homothety. Of the 81 students who solved the activity, 11 students outlined difficulties in the use of homothety. Five were blocked and changed their procedures to succeed. If most of them referred to homothety viewed as a specific geometric transformation, others seemed to have a general and figural approach of the problem based on the intuitive idea of picture enlargement.

In the first approach, the solution involves an axiomatic logic of geometry. We should point out that the results do not arise from perception, but from the knowledge of internal logic in geometry (it is necessary articulate figural and discursive reasoning geneses). The stress is now laid on the invariant linked to the geometric transformations regardless of the direct work on the figure. Entry through the transformations is normal in Geometry II, but only for exploring and verifying the properties of the configurations. The students are unskilled in construction, as it is irrelevant to the specific technique of

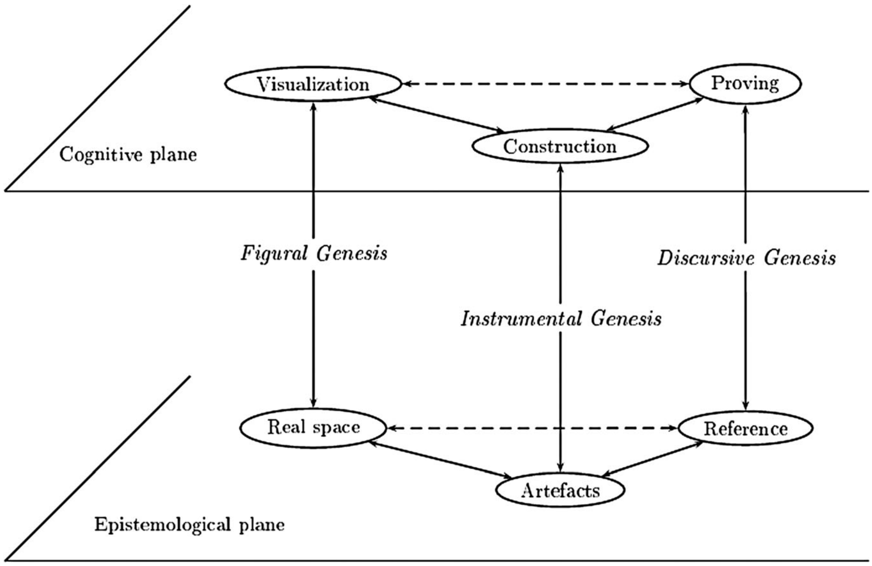


Figure 5. A geometric work switching to discursive genesis through instrumental genesis

their geometric knowledge. So they will have trouble transiting to instrumental approach.

Students of the second group focused on instrumental genesis through the search of commands related to enlargement and they found the tool "dilata" which they were exploring with lack of success. Some of them were evolving to a discursive genesis by thinking about homothety with ratio and searched it on a paper. In this case a global geometric work was observed which is described within Fig. 5.

Again, as in the former section, the mastering of the software influences the transition towards a discursive genesis. The transition to a complete geometric work depends on their knowledge about DGS. In the following, we focus an example related to this remark about figural and instrumental exploration leading to a solution. Daniel's solution can be considered representative of student solutions that use the software with ease and are not held up in their investigations. In his resolution process, Daniel uses the enlargement tools of the software and the logic of his construction is based on a paper and pencil approach. The drawing of the enlarged bell with the software is founded on the equilateral triangle which allowed the identification of angles. The figural genesis played a first-order heuristic

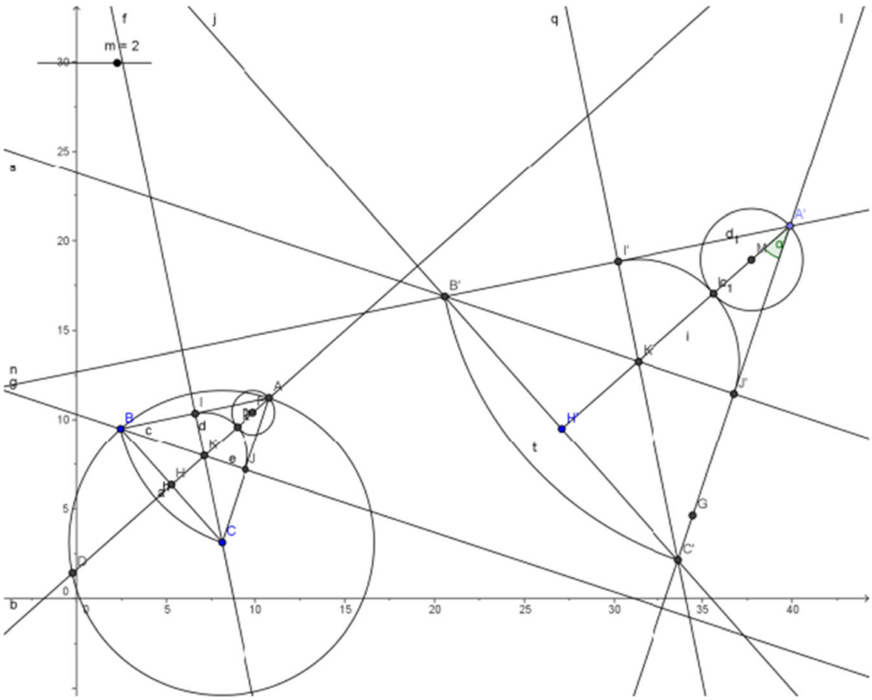


Figure 6. Daniel's solution

role for solving the problem with the software. This student shows his ease with the software through an elaborate solution using the sliders: “To enlarge the drawing with the given ratio, ratio $r = d\ (a, b)$, I used a slider m with initial value r and a segment ($A'H'$) with length $m.r$ (so I was able to modify the drawing if necessary). Then, I drew a line forming an angle of 30 ° with the segment, and then its symmetric and perpendicular lines to the segment. Thus, I obtained the triangle and the rest was done as previously (Fig. 6).

During the interview, Daniel assesses the process:

> GeoGebra works with geometric objects in a very direct manner, even more than with a ruler and compass. For me, geometric objects and their properties are inseparable. Maybe that’s why I find it so easy.... The program’s flexibility allows us to visualize relationships between figures and properties. For instance, the sliders are useful for parameterization and to see as objects vary and what remains constant. When I use the software I realize that I need to plan a priori a process using pencil and paper and I have to be clear on properties what I will use for doing it. With Geogebra, I can correct easily and once the problem is solved I can deepen my analysis and investigate properties and check a posteriori.

Daniel’s procedure illustrates the ability of some students to move easily between various geneses and geometric methods but unfortunately other face difficulties and need a strong teacher’s help to overcome their blockage.

Conclusion and Discussion

The primary aim of this study was to ascertain how dynamic geometry environments may influence and relate to geometric work. Based on the Space for Geometric Work framework, it focused on the three geneses (figural, instrumental, and discursive) involved in SGW. Even if the geometric task is simple and only dedicated to a construction, the data show a wide diversity of approaches among students, coming from variations in both their relationships with computers and software, and also their relationship, with geometry. Despite this diversity, two main forms of SGW have been observed. The first, Fig. 2, referring implicitly to Geometry II focuses mainly on a discursive genesis of proof supported by an analytic visual work on the figure with no specific attention paid to drawing tools. In this case, a point that can be underlined is the incompleteness of the global geometric work among a part of students who relied on figural and discursive aspects (hypothesis 1). Returning to the instrumental genesis can be problematic when there is no congruence, in Duval’s sense, between the theoretical tool and computer tool.

Geometric work of students that have not faced these difficulties is supported by an appropriate figural genesis in relationship with the artifact: these students made a semiotic treatment in a structured way and the corresponding instrumental deconstruction. Geometric reasoning often requires, in parallel, mobilizing two registers of representation, so that conversions are required continuously, although sometimes they are implicit. Our data show that possibilities offered by the artifact in its dual relationship with personal meanings and mathematicians become a semiotic potential of the artifact for SGW. However, this potential is not spontaneously active for most prospective teachers in the group.

The second SGW, Fig. 3, promotes instrumental genesis articulated with figural genesis (hypothesis 2). This result is not surprising, because of the task's emphasis of the construction of an object, though the students do show a good adaptation to instruments. However, we can note the special importance of a Geometry I work focusing on instrumental genesis among the students. Indeed, half of the student teachers built both the original and the enlarged bell with a construction based on angles. In this case, they identified the properties of symmetry and invariance of angles, and were guided by perception and visualization of the figure, as well as by use of instruments. As a result, they remained in plane (Fig–Ins) and they could use the same software commands for both tasks. Some difficulties can occur when the relationship between the two bells is to be established with the software. To interpret this impressive choice SGW guided by Geometry I, we need to consider the context of the task (what Brousseau calls the *milieu*), which provides very high performing construction tools such as GeoGebra. The institutional context, training teachers for teaching at secondary schools, could have also influenced the student teachers who tried to find elementary solutions. The task should also allow student teachers to see the difficulties they will have in explaining geometric properties to their own future students when engaged in construction problems with software. We noticed that there are gaps in the personal space of these future teachers when they have to relate the software to a mathematical meaning even if fifty percent of the students claimed that it was their explicit knowledge and previous study of geometric properties which guided their work with GeoGebra.

Another point to bear in mind relates to the ability of some students to switch to discursive genesis when experiencing blockage due to unsuccessful software exploration. In this case, the geometric work they made is complete but in another sense than in the first SGW based on discursive genesis (plane Dis–Fig). As a rule, teacher trainer assistance is needed to help students surmount their block and orient them to discursive genesis or furnish hints about software potentialities. As this study forms part of a task development project, this point was crucial to underscore with students aiming to become teacher. In this example, we used personal teachers' students SGW to develop a good "appropriate" SGW in the classroom and to describe the mathematical knowledge necessary for teaching. We also defined some aspects of cognitive and epistemological planes involved in the teaching process. One may also consider the distance between these two planes. For instance, when an artifact is being instrumentalized, action schemes are necessary; they enhance its development and its availability in the cognitive plane. What process is involved in converting flexible movement between various mathematical registers, including those of a visual nature in technology, and thus thwarting the phenomenon of compartmentalization in this context? In the training we aim to have the future teachers become aware of these three types of geneses that influence geometric work.

In a more theoretical perspective, it also seems that we could further clarify the relationship between the instrumental approach and the approach with SGW. We believe that adopting the more holistic perspective of SGW introduced a more complete and dynamic dimension to the global work of the teacher, as well as an element to the teacher orchestration generally seen as focusing on instrumental genesis. Our study shows that a perspective on the development of geometric reasoning must be added keeping the construction of a discursive genesis linked with elements of visual deconstruction as a strategic knowledge. This reasoning requires in particular a reflection on the role of properties and the techniques of proof validated by students in the developing process of a form of geometry directed by practical (Geometry I) or more axiomatic (Geometry II) goals. The concept of SGW could give the teachers and the teacher educator a framework to see the origin of properties and how they fit into the students' geometrical thinking along with instrument use and figure visualization. We also hope that introducing future teachers to these three types of geneses that influence geometric work can give them some distance from their own work and some geometric elements to structure their own learning and implement activities for their students in secondary education.

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Note

According to Rabardel (1996), the notion of artifact includes everything that has undergone a transformation, however small, of human origin. (…), Its significance is not restricted to material objects (physical world) but includes symbolic systems that can also be instruments.

References

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*(3), 245–274.

Coutat, S. (2006). *Intégration de la géométrie dynamique dans l'enseignement de la pour favoriser une liaison école-collège*. Grenoble: Thèse de l'université J. Fourier.

Coutat, S. & Richard, P. (2011). Les figures dynamiques dans un espace de travail mathématique pour l'apprentissage des propriétés géométriques. *Annales de Didactique et de Sciences Cognitives, 16*, 97–126.

Duval, R. (2005). Les conditions cognitives de l'apprentissage de la géométrie : développement de la visualisation, différenciation des raisonnements et coordination de leur fonctionnements. *Annales de Didactique et de Sciences Cognitives, 10*, 5–53.

Gomez-Chacon, I. & Kuzniak, A. (2011). Les Espaces de Travail Géométrique de futurs professeurs en contexte de connaissances technologiques et professionnelles. *Annales de didactique et de sciences cognitives, 16*, 187–216.

Houdement, C. & Kuzniak, A. (1999a). Un exemple de cadre conceptuel pour l'étude de l'enseignement de la géométrie en formation des maîtres. *Educational Studies in Mathematics, 40*(3), 283–312.

Houdement, C. & Kuzniak, A. (1999b). Géométrie et Paradigmes géométriques. *Petit x, n° 51*, 5–21.

Kuhn, T. S. (1966). *The structure of scientific revolutions* (2nd ed.). Chicago: University of Chicago Press.

Kuzniak, A. (2006). Paradigmes et espaces de travail géométriques. Éléments d'un cadre théorique pour l'enseignement et la formation des enseignants en géométrie. *Canadian Journal of Science and Mathematics Education, 6.2*, 167–188.

Kuzniak, A. (2011). L'espace de Travail Mathématique et ses genèses. *Annales de didactique et de sciences cognitives, 16*, 75–96.

Kuzniak, A. & Rauscher, J. C. (2011). How do teachers' approaches on geometrical work relate to geometry students learning difficulties? *Educational Studies in Mathematics, 77*(1), 129–147.

Laborde, C. (2001). Integration of technology in the design of geometry tasks with cabrigeometry. *International Journal of Computers for Mathematical Learning, 6*(3), 283–317.

Lagrange, J. B. (Ed.). (2009). *Genèses d'Usages Professionnels des Technologies chez les Enseignants. GUPTEn Rapport final Septembre 2009*. Paris: Université Paris-Diderot.

Mithalal, J. (2010). *Déconstruction instrumentale et déconstruction dimensionnelle dans le contexte de la géométrie dynamique tridimensionnelle*. Grenoble: Thèse de l'université J. Fourier.

Rabardel, P. (1995). *Les hommes et les technologies. Une approche cognitive des instruments contemporains* (p. 8). Paris: Université Paris.

Trouche, L. (2000). La parabole du gaucher et de la casserole à bec verseur : Etude des processus d'apprentissage dans un environnement de calculatrices symboliques. *Educational Studies in Mathematics, 41*, 239–264.

Yerushalmy, M. & Varda, T. (2004). Understanding dynamic behavior: Parent–child relations in dynamic geometry environments. *Educational Studies in Mathematics, 57*(1), 91–119.

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