Teaching and Learning Geometry and Beyond ...

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The always new question on the usefulness of maths and geometry learning

An essay of Arbuthnot in 1701

1. To develop Mind and Reasoning
   "Truth is the same thing to the Understanding as Music to the Ear and Beauty to the Eye"
2. For their applications in a wide variety of fields (Trade, Navigation, Art of War...)
3. To learn how to reach the results and not only the results. A method of freeing the mind from superstition.

À bas Euclide – Euclid must go ?

Some conflicting points of view

1. Modern mathematics : Dieudonné's provoking slogan paying attention to the distance existing between traditional triangle geometry based and contemporary geometry.
2. Counter-reformation : Down with Euclid who does not give the royal route to the applications and use of geometry in the real world.
3. Euclid as the first didactician...
4. A British viewpoint : The cry "Euclid must go !" has gained a certain notoriety in recent years. Our reaction to this in England was merely mild surprise since as far as we were concerned Euclid had already been gone for a long time. (Fletcher, ESM. 3-3 1971)

Developping teaching geometry and researches in didactics of geometry

In a specific context

1. A tension between utilitarian and idealist visions
2. Using new drawing tools which change discovering and proving methods
3. Towards a new “geometrical literacy” revisiting the relations with Truth and Proof among citizens

Conferences of European Research in Mathematics Education

1. Development of spatial abilities and geometrical thinking through consecutive educational levels.
2. Geometry education and the “real world” : geometrization and applications
3. Instrumentation : artefacts such as, computers and the way they are used
4. Explanation, argumentation and proof in geometry education.
5. Some theoretical aspects : Van Hiele’s levels ; Registers of semiotic representation ; geometrical paradigms.

Developping teaching geometry and researches in didactics of geometry

Shaping a theoretical framework dedicated

1. To study the teaching and learning of elementary geometry
   - During compulsory education
   - During teacher training
2. To compare the teaching of geometry in different institutions and countries
3. Focused on the “geometric work” considered a crucial point

Mathematic work as a crucial topic of study

Freudenthal's view

What is mathematics ? Of course you know that mathematics is an activity because you are active mathematicians. It is an activity of solving problems, of looking for problems, but it is also an activity of organizing a subject matter. ESM 3 - 1971

Thurston's conception

Mathematics includes integers numbers and geometry plane and solids
Mathematics is that Mathematicians study
Mathematicians are those human beings who make advance human understanding of mathematics.

The mathematic work, activity and teaching

Mathematic work and activity : Habermas

By *work* or rational activity relative to an end, I hear or an instrumental activity, or a rational choice, or else a combination of both

Freudenthal ESM 3 1971

A great part of mathematical activity today is organizing. We like to offer the results of our mathematical activity in a well organized form where no traces betray the activity by which they were created. This objectivation is a habit of mathematicians from the oldest times. It is a good habit, and it is a bad one. We freeze up the result of our activity into a rigid system, because this is objective, because it is rational, and because it is beautiful,and this we teach.

Contexts and phases of work

Work context : Reichenbach

1. Context of discovery
2. Context of justification
3. *Context of usage*

Phases of work

1. Making the discovery
2. Presenting the result
3. Becoming familiar with the results

A Space for Geometric Work

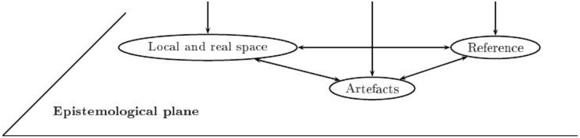
A Space for Geometric Work (SGW) is a place organized to ensure the geometric work (in an educational context).
Based on two dimensions

- An epistemologic level
- A Cognitive level

The epistemologic level

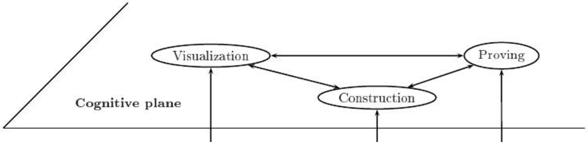
A network of the three following components :

- A real and local space as material support with a set of concrete and tangible objects
- A set of artefacts such as drawing instruments or software
- A theoretical frame of reference based on definitions and properties

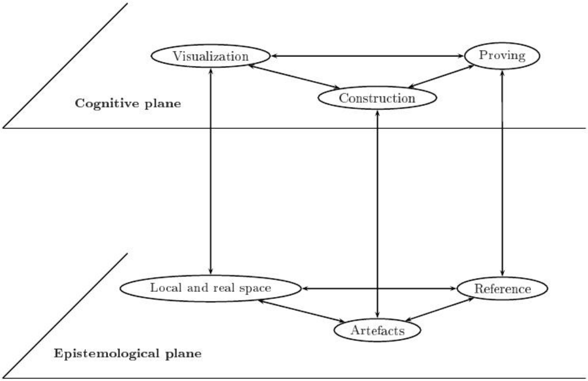


The cognitive level

- A visualisation process connected to the representation of space and material support
- A construction process determined by the instruments (rules, compass, etc.) and geometric configurations
- A discursive process which conveys reasoning and proof



How to organize each level ?
How to articulate both levels ?
What guides the work ?

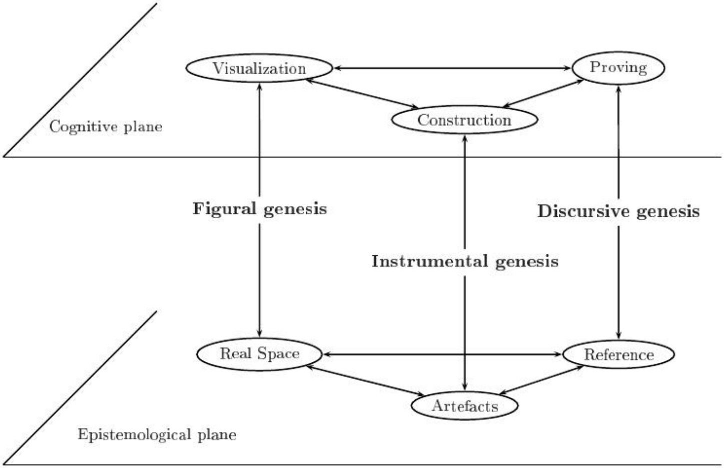


Various geneses

Both levels, cognitive and epistemological, need to be articulated in order to ensure a coherent and complete geometric work. This process supposes some transformations that it is possible to pinpoint through three fundamental geneses strictly related to our first diagram :

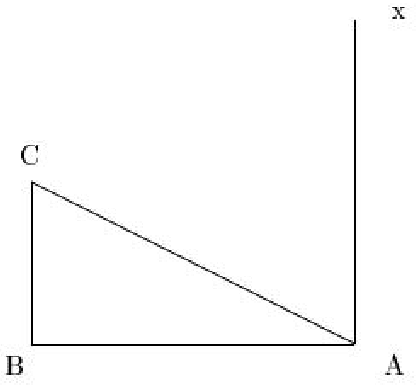
- An instrumental genesis which transforms artefacts in tools within the construction process.
- A figural and semiotic genesis which provides the tangible objects their status of operating mathematical objects.
- A discursive genesis of proof which gives a meaning to properties used within mathematical reasoning.

A general view on Space for Geometric Work



What guides the work : In search of the paradigms

Let ABC be a triangle with a right angle in B, with $AB = 4cm$ and $BC = 2cm$. The ray (Ax) is perpendicular to the line (AB). And M is a point on the ray (Ax). The purpose of this problem is to obtain particular configurations of the triangle AMC.



Question : Does a point M exist such that the triangle ACM is equilateral ? Justify your reply

A student's answer

The correct answer is “no” and it can be shown, using a compass, that there is no third vertex on the ray (Ax) for the equilateral triangle constructed on the side (AC).

A response emblematic of Geometry I

A student carries out an experiment in the real, perceptible world by constructing a triangle with drawing instruments and then s/he realises that no crossing points lie on the line where one point should be for the triangle to be equilateral.

Answer expected in French Secondary School

If ACM is an equilateral triangle with M on Ax, the angle $\widehat{MAC}$ will measure 60° and the angle $\widehat{CAB}$ 30° (sum of the three angles of a triangle) and by symmetry the angle $\widehat{CAC'}$ will be 60° (C' is the symmetric of C through the line (AB)). As the triangle CAC' is isosceles in A (by symmetry), it should be equilateral.
This is not true because the length of $C'C$ is 4, which is unequal to CA and $C'A$ ($2.\sqrt{5}$ by the Pythagorean theorem).

An answer given by a Student Teacher

We can explain this by the fact that in an equilateral triangle all the angles are equals and the sum of the angles is 180°. The value of each is 60° . In this case, when we measure with a protractor, we observe that $\widehat{CAM}$ is more than 60°, indead $\widehat{CAM} = 64^\circ$.

The example highlights two main points

- There exist various viewpoints on elementary geometry, which can result in conflicting solutions to the same problem
- Certain terms (such as construction) can assume different meanings depending on teachers' and students' standpoints on the same question

The notion of geometrical paradigm will be useful to understand, clarify and organise these various points of view.

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CERME3 : Houdement and Kuzniak

Geometrical paradigms

Two working hypothesis

1. In education, the sole term geometry evokes several distinct paradigms. By and large, these paradigms reflect the breaks observed between the various academic cycles in the teaching and learning of geometry.
2. Students (pre-service teachers), teachers and school students are working in distinct paradigms : this epistemological difference explains some didactic misunderstandings.

Geometrical paradigms

In search of geometrical paradigms

With the notion of paradigms, Kuhn has enlarged the idea of a theory to include the members of a community who share a common theory.
A paradigm is what the members of a scientific community share, and, a scientific community consists of men who share a paradigm p. 180. (Kuhn The structure of scientific revolutions 1962 then 1966)

Geometrical paradigms

Two meanings

1. In its most global use, the term paradigm stands for the entire constellation of beliefs, values, techniques, and so on shared by the members of a given community.
2. On the other, it denotes one sort of element in that constellation, the concrete problem-solutions which, employed as models or examples, can replace explicit rules as basis for the solution of the remaining problem of normal science.

Three Elementary Geometries

Geometry I Natural Geometry
or the confusion between geometry and reality

Geometry II Natural Axiomatic Geometry
or Geometry as a schema of reality

Geometry III Formal Axiomatic Geometry
or the independency of geometry and reality

Geometry I

- A geometry involved in the real world.
- From the reality to drawings and drawings as real objects.
- The question of approximation and measurement

Geometry II

- A geometry closely linked to the real world from which it is a model
- A reasonning based on schemas of reality
- Axiomatization as an horizon

Geometry III

- The question of axiomatic coherency and completion axiomatique
- The question of the axiomatic organization
- The break between truth and certitude

Geometrical paradigms

These geometries are not ranked. Their horizons are different and so the nature and the handling of problems change

- Geometry I Technological and practical
- Geometry II Axiomatic and modeling
- Geometry III Logic and formal

Diversity of Spaces for Geometric Work

The reference SGW, The expected reorganization

This working space is normally defined and based on mathematical criteria. But also it depends on social, economical and political criteria.
Study based on Treatises, Curriculum
Epistemological vigilance

Diversity of Spaces for Geometric Work

Personal SGW

When the problem is set a person (the pupil, the student or the professor), either an ideal expert, this person handles the problem with its personal SGW.
The cognitive plane is concerned by a specific individual and not an epistemic or institutional subject.
Conceptions, knowledge of teachers and students :
the **Cognitive vigilance**

Diversity of Spaces for Geometric Work

The implemented SGW or the didactising question

When the general paradigm is accepted and the reference SGW is built, it remains to teach geometry to students and for that it's necessary to organize a suitable SGW to convey the kind of geometry expected by the educational institution.
The geometrical working space turns to be suitable only if it allows to the user to link and master the three components defining the working space. Curriculum, textbooks, real class implementation
the **Didactic vigilance**

Today implemented SGW in France

The case of the inscribed angles at Grade 9

1. Two main properties Eucl III.20 and III.21
2. Rotations, angles, regular polygons.

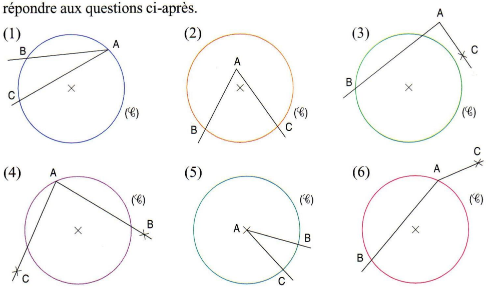
From a textbook Grade 9

On figure 5, the angle $\widehat{BAC}$ is a central angle. It is not the case of $\widehat{BAC}$ on the other figures.

Angle inscrit et angle au centre

7. Des cercles et des angles *exercices 10 à 12 p. 171 et 172*

a/ Observer la disposition de l'angle $\widehat{BAC}$ sur chacune des figures ci-dessous puis répondre aux questions ci-après.



Deduce from this which characteristics a central angle seems have.

Fundamental properties based on measurement

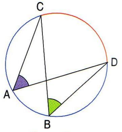
Draw a circle with center O. Draw several inscribed angles in the circle which intercept the same arc BC. Measure these angles. What can you conclude ?
Draw a circle with center O. Draw a central angle and an inscribed angle which intercept the same arc BC. Measure both angles. Do it the same several times. What can you conclude ?(p. 165)

Fundamental properties based on measurement

d/ Propriétés

Si deux angles inscrits dans un cercle interceptent le même arc alors ils ont la même mesure.

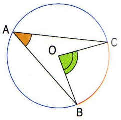
Sur le dessin ci-contre, $\widehat{CAD}$ et $\widehat{CBD}$ sont deux angles inscrits qui interceptent le même arc $\overset{\frown}{CD}$.



$\widehat{CAD} = \widehat{CBD}$

Si, dans un cercle, un angle au centre et un angle inscrit interceptent le même arc alors la mesure de l'angle au centre est le double de la mesure de l'angle inscrit.

Sur le dessin ci-contre, l'angle inscrit $\widehat{BAC}$ et l'angle au centre $\widehat{BOC}$ interceptent le même arc $\overset{\frown}{BC}$ donc $\widehat{BOC} = 2\,\widehat{BAC}$.



O est le centre du cercle.

Some characteristics

1. A practice based on “abductive“ way to introduce definitions and properties too.
2. Use of measurement
3. A series of properties not necessarily linked
4. A reasoning generally based on a particular (not generic) figure

Which is finally the implemented SGW ? We will speak of a Fragmented Geometry (I / II)

An example of coherent Geometry I

Alfonso has just come from a journey in the pre-cordillera where he saw a field with a quadrilateral shape which has interested his family. We want to estimate its area. For that, during it journey, he measured, successively, the four sides of the field and he found, approximatively : 300 m, 900 m, 610 m, 440 m. Yet, he does not reach to find the area.
Working with your classmates, could you help Alfonso to determine the area of the field.

An example of coherent Geometry I

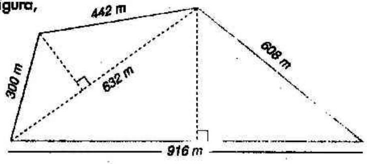
The exercise is then completed by the following hint : We can say you that, when you were working, Alfonso has explained its problem to her friend Rayen and she asked him to take another length of the field : a diagonal.
Alfonso has come back with the data : 630 m.
Has it done right ? Could we help him now if we could not do it before ?

Decomposition of the figure and calculation after measuring on the figure

¿Puedes estimar el área de la parcela de la figura, a partir de las mediciones indicadas?

Solución: Podemos descomponer la parcela en pedazos triangulares como los indicados y reconstruir estos triángulos a partir de las mediciones tomadas.

¿Cómo calculamos ahora el área?



How can we compute the area now ? Well, we determine the scale of the drawing, we measure the indicated height and we obtain the area of each triangle (by multiplying each length of a base by the half of the corresponding height).

Toward a work on approximation in Geometry

We can tell you that his friend Horacio has found 130 000 m^2, approximatively. When Rayen have heard that. She said, it is no possible ! It is the double.
Could yo find the result with any calculation ?

Te podemos contar que a tu compañero Horacio, le dio 130.000 m^2, aproximadamente, es decir 13 hectáreas. Cuando Rayén escuchó esto, dijo: ¡No puede ser! ¡Es como el doble de eso!

¿Serías capaz, como Rayén, de estimar "a simple vista" el área total?

A nosotros nos dio un área total de 240.600 m^2 o 24,6 hectáreas, aproximadamente. ¿Y a ti?

A work on the approximation closely related to a Geometry based on the possibility of measuring

The impact of DGS on Spaces for Geometric Work // Before the massive use of software

1. Constructing a particular figure with drawing tools
2. Measuring on the figures with instruments
3. Drawing a conjecture
4. Institutionalisation of the property which is admited or proved after

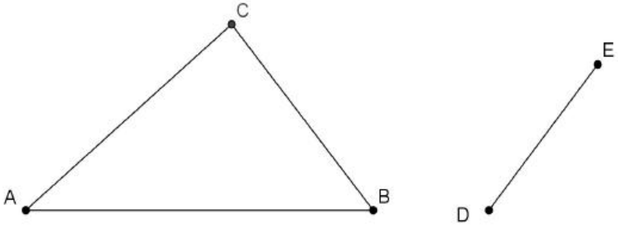
The impact of DGS on the Geometric Work Space

1. Constructing a figure with DGS
2. Measurement given by the software
3. Dragging of some points to see that the property rests true
4. Institutionalisation of the property which is admited or proved after

Similar triangles in an ordinary Grade 10 class

Create a triangle DEF such that $\widehat{BAC} = \widehat{EDF}$ and $\widehat{ABC} = \widehat{DEF}$

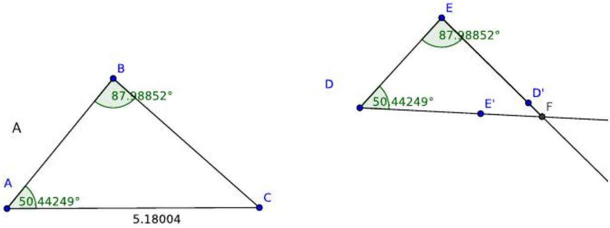
Partie I : Créer un triangle DEF tel que $\widehat{BAC} = \widehat{EDF}$, $\widehat{ABC} = \widehat{DEF}$.



- What can we say about $\widehat{ACB}$ et $\widehat{DFE}$
- With your ruler, compare the sides of the triangle ; What do you observe ?
- Complete the text : We can conjecture that if two triangles have ... then theirs side are ...

Similar triangles in an ordinary classroom
With a software

Create a triangle DEF such that $\widehat{BAC} = \widehat{EDF}$ and $\widehat{ABC} = \widehat{DEF}$



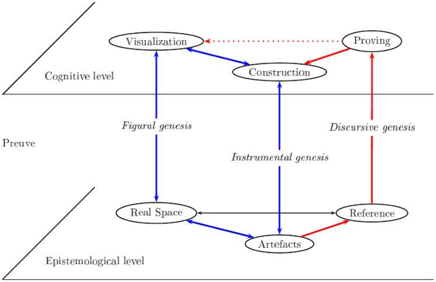
The general misunderstanding or When the monstration becomes a demonstration

- Teacher : Do we have demonstrate the property ?
- Students : Yes we have made a demonstration
- Teacher : No, it's too imprecise

The break between two approaches of Geometry

For the teacher

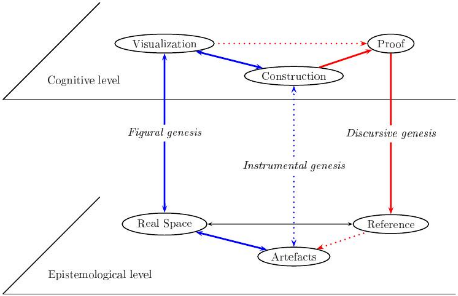
- The construction is simple and will not cause trouble
- Motivating the entrance in Geometry II by a first work in Geometry I : to motivate the formal proof
- A drawing seen as a generic figure



A break with the personal SGW of the students

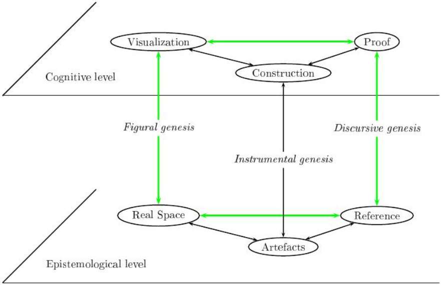
For them

- To construct with drawing tools is complex and need a long time
- Due to a diversity of results, a great diversity of properties are drawn by the students
- A particular figure without any generalisation



Misunderstanding and didactical contract

- The work of construction made by students is ignored by the teacher
- The sofware as a source of Truth
- The conjecture as a basis of the agreement.
- Experimental proof and Axiomatic based proof



Misunderstanding and didactical contract

A subreptice shift to Geometry I

- The reference SGW always insists on a transition to GII based on GI
- The standard implemented SGW is unstable and dependant of the level of the students
- A real shift to Geometry I favoured by the software which gives the “proof”
- The lack of a theoretic system to ground the teaching
- The reorganisation of the SGW is only led by the teacher who tries to adapt its teaching to the supposed low student’s level.
- A lack of epistemological vigilance conducts to a lost of cognitive vigilance

Some perspectives

- Maintaining hope
- Building a coherent and global SGW depending on the institution
- Towards a rich Geometry I with a SGW well structured around (GI/GII)
  - Not fragmented and related to other fields
  - With a work and approximation
- Thinking beyond geometry to reach a real geometric culture

Researches

- Geometry and other areas (in maths)
  - Geometry and Numbers : Real and Complex numbers and geometrisation.
  - Modelling problem based on geometric situations with use of software –> Analisis, functions and proof...
- On and with the theoretical framework
  - About the various geneses.
  - Articulation of areas and registers : Space for Mathematic Work (ETM)
  - Welcome in Madrid for the symposium ETM4

Merci
Çok teşekkür