# Spaces for Mathematical Work: Viewpoints and perspectives

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This special issue of *Revista Latinoamericana de Investigación en Matemática Educativa* (RELIME) is dedicated entirely to the articles proposed within the framework of the third Symposium Mathematical Work Space (ETM, in French) which is devoted to the study, development and possible applications of the ETM concept in the didactics of mathematics. The mathematical work and its functioning within the school setting are the foundation of the ETM approach, and in this introduction, before presenting the thematic organization of the contributions; we will summarize this theoretical approach. Its goal is to enrich, in a non-normative manner, the didactic study of the mathematical work of the students and teachers.

## 1 A didactic perspective on mathematical work

The teaching of mathematics has been subject to a profound and worldwide re-examination since the beginning of the 1960s. Before this period the citizens of developed countries were offered two types of teaching by the school systems. The children with working class background were offered a brief and essentially utilitarian training in elementary mathematics. The children of the bourgeoisie, whose destiny was to take the economy and the governing of the country into their own hands, were given the opportunity to have access to a teaching of mathematics that was regarded as schooling in logical reasoning. In both cases, the student was placed in the position of a recipient of knowledge dispensed by the teachers. Starting from the 1960s, various phenomena have contributed to a change in the views on the role and the way of teaching of mathematics. Without pretending to be exhaustive, one can mention the reform of modern mathematics, the development of the research on the learning experience of children, or also, the massification of the education in a context of economic and ideological competition.

For the purpose of our discussion, we will retain two profound characteristics of the changes which have become accentuated since: on the one hand, bringing to light the diversity of the work of the mathematician, viewed as the main actor in the advance of mathematics, and on the other hand, the

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pedagogical idea of stimulating the student's activity in order to enhance the student's ability to develop his or her knowledge in a problem-solving context. In each case, the mathematical work is in the core of the evolution, which leads logically to placing this concept centre stage in the didactics of mathematics.

The work we are referring to is a rational activity that is oriented towards a specific goal and is able to rely or not on the use of a certain number of specific instruments and artefacts. In mathematics, the purpose of this activity must be centred on the objects studied by the mathematicians, "these human beings working to advance the human understanding of mathematics" (Thurston, 1994, p. 162). Therefore, we consider that the didactic research should be concentrated on this double point of view – the learning experience of the students and the organization of this learning experience by the teacher within the context of a schooling that favours the development of the mathematical work of the student.

## ❷ The Space for Mathematical Work concept

The general Space for Mathematical Work concept (ETM) expands the work space concept in geometry, introduced by Kuzniak and Houdement (Kuzniak, 2006) in the study of the didactics in this area. It was developed in order to help for better understand the didactical challenges around the mathematical work in a school environment. The space, conceived in this way, designs a well-thought and organized environment in order to permit the work of the individual solving mathematical problems. In the case of school mathematics, these individuals, generally, are not experts but pupils or students, either experienced or beginners.

From the specific study of geometry we retain the principle for articulation of two levels in the ETM (Kuzniak, 2011): the first one of epistemological nature, in a close relation with the mathematical content of the studied area, and the other of cognitive nature, related to the thinking of the person solving mathematical tasks.

Therefore, the mathematical work is derived from a process which permits to impart, on the one hand, progressively gives a meaning to each of the epistemological and cognitive levels and, on the other hand, formulates these two levels thanks to different geneses. In the case of geometry, the entire process could be described as proceeding from the elements of the following diagram (Fig. 1), but it would need certain modifications in order to adapt it to the general framework of the ETM:

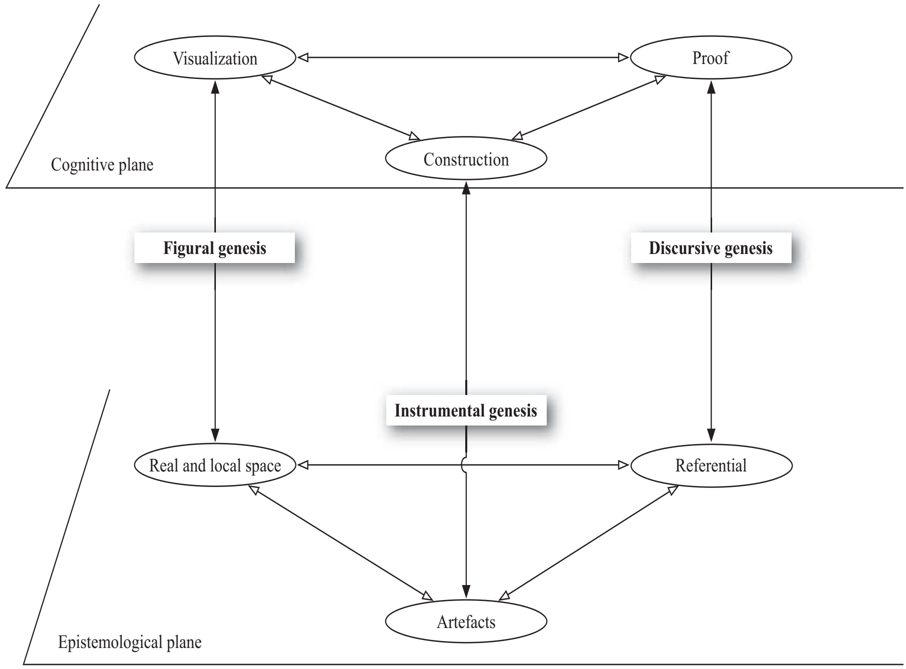


*Figure 1.* The Space for Geometric Workspace and its geneses

## 2.1. *The epistemological level and its components*

As regards the geometry, three components in interaction are characteristic for the geometric activity in its purely mathematical dimension:

- a real and local space as support material, with one set of concrete and tangibles objects;
- a set of artefacts such as drawing instruments or software;
- a theoretical reference system based on definitions and properties.

These components are not juxtaposed; they must be organized according to a predetermined goal which will depend on the mathematical domain in its epistemological dimension. While the emphasis is on the process of the learning of the student in a didactic situation, this epistemological plan can be considered also as an *epistemological environment* (Coutat & Richard, 2011).

If the artefacts and the theoretical referential remain two basic components of any epistemological plan associated with a specific mathematical domain, the component linked to the space and to the geometrical configurations must be modified in order to be expanded to other mathematical domains. In accordance with one concept of the mathematics based on semiotic representations, which goes beyond the single consideration of systems of representation, it seems pertinent to use the term sign or *representamen,* in the sense of Peirce. Therefore, the sign or the *representamen* is "something" which represents something else that can be its object or maybe also itself as such. Following the respective mathematical domain, the signs can be geometrical images, algebraic symbols or graphics and even tokens; they can also be models or photos in the case of problems that involve modelling. Unlike the signs with dyadic structure, which retain only the relation of reference between the signifier and the represented object, the idea of a sign, which is also its own representation, invites to revisiting the semiotic process when the mathematical work is at stake. This is particularly visible when a geometrical image, which is a form in itself, is at the same time *representamen* and model of representation (Coutat, Laborde & Richard, 2013).

## 2.2. *The cognitive process level*

The mathematics that is taught is not a disembodied set of properties and objects reduced to signifiers that can be manipulated by formal systems – it is first of all and mainly a human activity. Therefore, it is essential to understand how communities of individuals, but also specific individuals, use and internalize the mathematical knowledge in their practice of the discipline. It is also essential to understand how they will impart a meaning to all these tangible signs and objects. This implies a second level of the ETM centred on the subject which is viewed as a cognitive subject. This introduction to the cognitive field must be made in close relation to the components of the epistemological level and, in order to remain within the didactical framework, it is possible to adapt the semiotic approach of Duval (1995, 2005). For the geometrical activity, these processes are as follows:

- a process of visualization related to the representation of the space and the support material;
- a process of construction and function of the used instruments (rulers, compass, etc.) and the respective geometrical configurations;
- a discursive process producing arguments and proves.

The process of visualization needs to be specified precisely in order to find its place in an extension to the ETM. It must be associated with the diagrams and operations of the use of the signs, about which nothing proves *a priori* that they all pick up the entire visualization as such, even within its extended conceptualization (Fig.2).

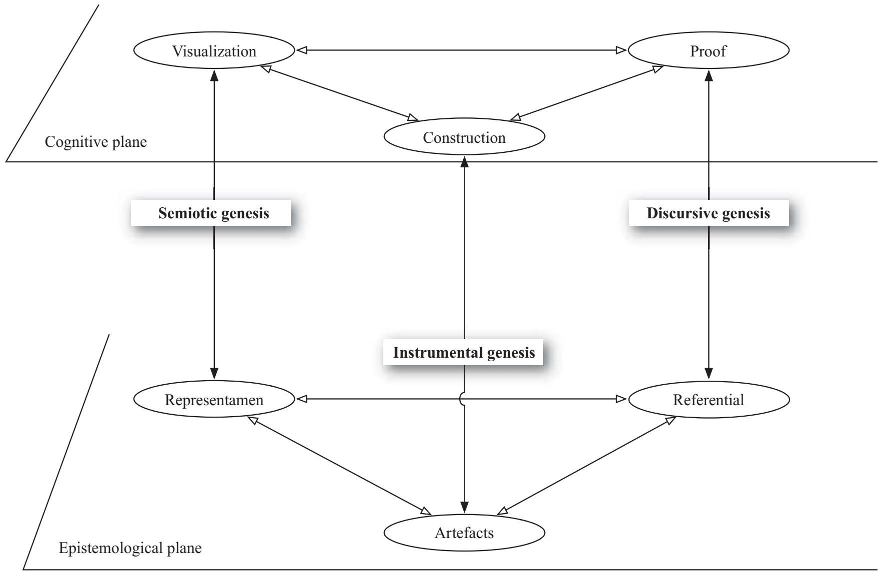


*Figure 2.* The Space for Mathematical Work and its geneses

This process of "extended" visualization has to be well distinguished from the simple vision or perception of the objects; it can be envisaged as a process of structuring the information provided by the diagrams and the signs. It nourishes the intuition of the properties and sometimes contributes to establishing cognitively the validity of these properties. Under certain conditions, it can be likened to a reasoning of a discursive-graphical type (Richard, 2004) and can be expressed in the interior of the registers of a determined semiotic representation.

## ❸ The reference, adequate and personal ETM

Within the theoretical framework of ETM, just as within that of Space for Geometrical Work (ETG), the paradigm concept orients and structures the organization of the components which, due to their different functions, participate in the specifics of the different paradigms at play. A paradigm is instituted when a given community of individuals agree to formulate problems and organize their solutions by prioritizing certain tools or certain forms of thinking. The "paradigmatic" workspace, such as it is defined by this community, will be called a *reference ETM*. In a given educational institution, the solving of a problem assumes that an *adequate ETM* could be organized in order to permit a student to participate in the solving of the problem. This adequate ETM must necessarily fulfil two conditions: on the one hand, to permit work in the paradigm that corresponds to the problems in question; on the other hand, to be "well designed" in the sense that its different components are organized in a valid manner. Its designer plays a role here that is similar to that of the architect who designs a workspace for the potential users. In the classroom, the design of this space will depend on the *personal ETM* of the teacher. When the problem is assigned to a student, its mathematical treatment by the student will take place within the *personal ETM* of this student. Based on this fact, the *adequate ETM* is not fixed and has to be continuously modified in order to adjust itself to the local constraints.

Therefore, the mathematical work within school settings can be described at three ETM levels: the mathematics as seen by the institution is described in the reference ETM; it must be converted by the teacher into adequate ETM to permit its effective implementation in class where everyone will work in his/her personal ETM; the choice and organization of the tasks given by the teacher to the students are essential in the constitution of the adequate ETM, so that the teacher gives an opportunity to the students to solve in an adequate manner the questions offered, i.e. in a way that is in conformity with the institutional expectations that are described more or less explicitly in the reference ETM. These choices and the management of the activities will depend, to a large extent, on the personal ETM of the teacher. The observation of the activity of the students will permit them to identify their personal ETMs by identifying eventual sub-assemblies of stable practices in them.

## 4 The geneses of the work in the ETM

The development by an individual of his or her mathematical work takes place gradually and passes through a gradual approach to the implementation of his or her personal ETM. This global genesis of the ETM supposes the use of a set of geneses which are interdependent and concern all epistemological components and the cognitive processes. The activation and the control of these geneses can be initiated by the teacher (at the level of the adequate ETM). It is important to know to what extent they are in conformity upstream with the expectations defined in the reference ETM (Kuzniak & Rauscher, 2011; Kuzniak, 2013).

As we see it, the epistemological and cognitive plans structure the ETM in two levels and help to understand the circulation of the knowledge within the mathematical work. How then, proceeding from here, should we formulate the epistemological and cognitive levels, in an operative manner, so that it is possible for the expected mathematical to work? We think that it is appropriate to use the three fundamental geneses as a base of the theoretical framework developed above:

- The Instrumental genesis, which allows us to make the artefacts operational in the constructive process contributing to the accomplishment of the mathematical work;
- The semiotic genesis based namely on the registers of the semiotic representation which gives meaning to the ETM objects and confers to them their status of operative mathematical objects. In this way, this semiotic genesis ensures the relationships between syntax, semantics, function and structure of the conveyed signs;
- The discursive genesis of the proof used by the properties combined together in the theoretical referential in order to put them in service to the mathematical reasoning and to a non-exclusively iconic, graphic or instrumented validation.

In order to define the geometrical work within the framework of the ETG, Coutat and Richard (2011) describe the interactions that are specific to the geometrical approach (see Fig. 4) by characterizing the three vertical plans which appear naturally in the diagram of the ETG. In our effort to provide a theoretical construct of an ETM by generalizing the achievements of the research on the ETGs, the vertical plans introduced in this way can be related to different phases of the mathematical work implemented within the execution of a given task: discovery and exploration, justification and reasoning, presentation and communication. The effective realization of these phases will define *de facto* a certain number of cognitive mathematical competences based on the coordination of the geneses in their relations with the epistemological plan (Fig. 3).

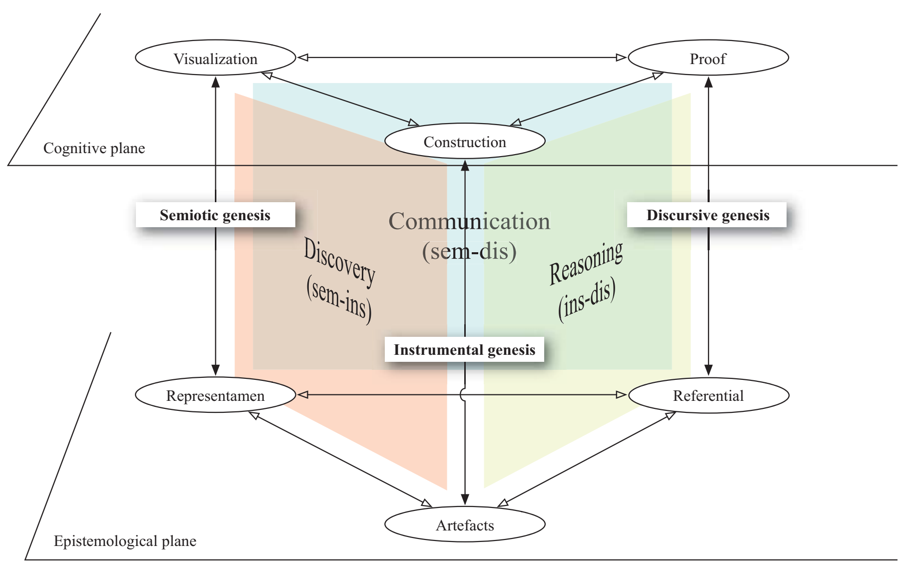


*Figure 3.* The vertical plans in the ETM

A first type of interaction favours the identification and exploration of objects on the basis of the semiotic and instrumental geneses in order to develop a competence related to the discovery of a solution of the mathematical problems. A second type of interaction develops the mathematical reasoning on the basis of justification of the discoveries and formulating the instrumental and discursive geneses. And finally, one last type is oriented towards the mathematical communication of the results and is based essentially on the semiotic and discursive geneses. The exact definition of these plans of interactions and the description of their interrelations depend on the specific mathematical domain that is studied.

## ❺ About the themes of this special issue

The articles in this issue are centred on the mathematical work and it constitutes the scientific core around which the cohesion of the scientific community was formed during the symposium. Without limiting themselves to the elaboration of a Space for Mathematical Work in its technical sense, the object of the articles retained for the special issue is devoted on a wider scale to the study of the semiotic, cognitive and instrumental dimensions of the mathematical work without excluding *a priori* any epistemological or didactic approaches. This expansion of the problematic to include the entire mathematical work in its different forms has been submitted to the wisdom of the contributors starting with four subjects for reflection, which can be found in the present issue.

### Theme 1 – *The mathematical work and the ETM*

The object of this theme is, on the one hand, to go deeper into the theoretical model defined by the Space for Mathematical Works and, on the other hand, to explore their use as analytical tool in specific studies. The use of this model in other domains than geometry supposes a specific and targeted study of the areas in question and suggests thinking about ETMs which, as in the geometry, can be applied to the algebra, the analysis, the arithmetic… In order to harmonize the notations, these specific workspaces, associated to the specific domains *d*, will be notated as $ETM_d$, i.e. $ETM_{algebra}$, $ETM_{analysis}$, $ETM_{arithmetic}$, … The Space for Mathematical Work can be viewed as a network of diverse fibres constituting the $ETM_d$. Therefore, the issue is knowing how the links between the spaces or the browsing of the plans are organized. These interactions between the domains are essential for understanding the global functioning of the mathematical work and, in addition, they require taking into consideration the processes of modelling within the framework of the ETMs, besides the purely semiotic issues.

### Theme 2 – *Technological environments and mathematical work*

This theme is interested specifically in the use of technological environments not for themselves, but to specify to what extent they affect the mathematical work. In fact, the emergence of many new contemporary instruments has contributed to the formation of a new relief of the constructive aspects and the artefacts supporting the execution of the mathematical work both at the level expected by the students and the level of the current research. We can consider their impact twice.

- In the first place, what are the potentials of such environments for transforming the mathematical work of the student? By taking into consideration the diverse geneses, it is advantageous to move out of the instrumental approach in order to answer this question.
- The second question requires studying in what aspects the use of technological environments influences the epistemological formation of the student, guiding his/her mathematical work. This can concern, for illustrative purposes only, the nature of the mathematical objects constructed and the acceptable mathematical proofs, as well as its role as a research instrument.

In order to answer this issue within the specific framework of geometry in the technological environments, Coutat and Richard (2011) have proposed to characterize the vertical plans of the adequate ETG by basing them on the approach idea: validation, modelling and discovery (Fig. 4). As regards the structuring of the ETM proposed above (Fig. 3), these approaches constitute the manifestations of the mathematical competences of the subject in the course of the geometrical work.

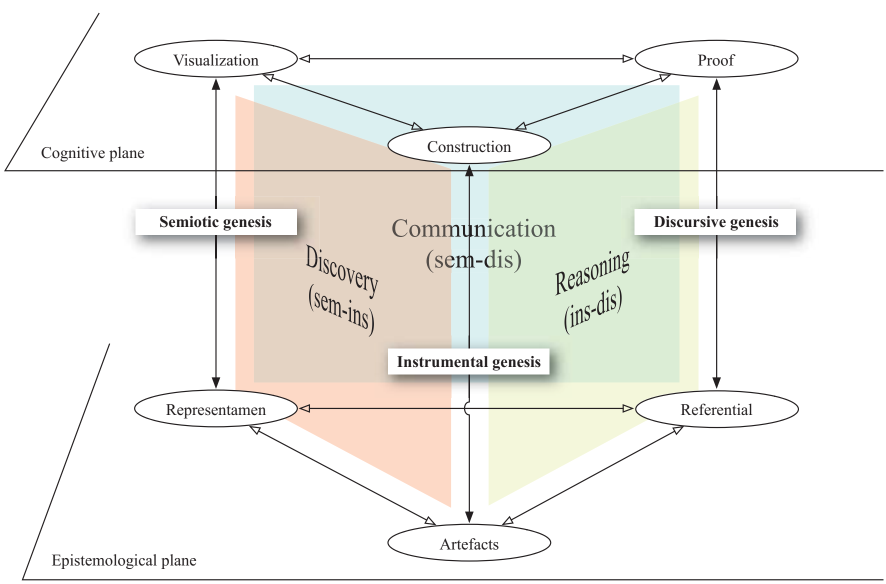


*Figure 4.* The vertical plans in the adequate ETG

Theme 3 – *The mathematical work and its social and institutional aspects*

Intrinsically, the question of the contexts is central in the constitution of the mathematical work. It can be about an internal approach in order to characterize two facets of the research work of the mathematician with the discovery and justification contexts. The context of use of the mathematics can be added to the preceding contexts, which is often the main context for the students and the regular users, who are generally interested in it because of the power of its applications. It is also possible to expand the view on the mathematical work by observing the role of the specific institutions, into which this work is inserted, along with the play of the social and linguistic interactions. The role of the education, the initial or continuous one, and of the mathematic teachers appears here as a fundamental institutional lever.

Theme 4 – *Visualization and representation in the mathematical work*

From the fact of the variety of the graphical representations used in all domains of mathematics, the question about the visualization and its global role in the mathematical work inevitably arises. While the visualization has been object of numerous research investigations in geometry, there are much fewer investigations on the visualization in other areas of mathematics, although many contemporary publications emphasize their importance (Guzmán, 1996; Alsina & Nelsen, 2006). This theme is interested in the topics of flexibility, in the genesis of the semiotic representation registers and, more generally, in the place of these registers in the mathematical work, traditional or instrumented.

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Thème 3 – *Le travail mathématique et les aspects sociaux et institutionnels*

De manière intrinsèque, la question des contextes est centrale dans la constitution du travail mathématique. Il peut s'agir d'une approche interne pour caractériser deux facettes du travail de recherche du mathématicien avec les contextes de découverte et de justification. Aux contextes précédents, on peut ajouter le contexte d'usage des mathématiques qui, souvent, est le contexte principal pour les élèves et les utilisateurs usuels des mathématiques, qui s'intéressent généralement à celles-ci pour la puissance de ses applications. Il est également possible d'élargir le regard sur le travail mathématique en observant le rôle des institutions particulières dans lesquelles ce travail s'insère, conjointement au jeu des interactions sociales et langagières. Le rôle de formation, initiale ou continue, des enseignants en mathématiques apparaît ici comme un levier institutionnel fondamental.

Thème 4 – *Visualisation et représentation dans le travail mathématique*

Du fait de la variété des représentations graphiques utilisées dans tous les domaines mathématiques, la question de la visualisation et de son rôle global dans le travail mathématique se pose inévitablement. Si la visualisation a fait l'objet de nombreux travaux en géométrie, beaucoup moins de recherches concernent la visualisation dans d'autres domaines mathématiques, encore que plusieurs publications contemporaines en soulignent l'importance (Guzmán, 1996 ; Alsina & Nelsen, 2006). Ce thème s'intéresse aux notions de flexibilité, à la genèse des registres de représentation sémiotique et plus généralement, à la place de ces registres dans le travail mathématique, traditionnel ou instrumenté.

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