Understanding the Nature of the Geometric Work Through Its Development and Its Transformations

Kuzniak Alain

Abstract The question of the teaching and learning of geometry has been profoundly renewed by the appearance of Dynamic Geometry Software (DGS). These new artefacts and tools have modified the nature of geometry by changing the methods of construction and validation. They also have profoundly altered the cognitive nature of student work, giving new meaning to visualisation and experimentation. In our presentation, we show how the study of some geneses (figural, instrumental and discursive) could clarify the transformation of geometric knowledge in school context. The argumentation is supported on the framework of Geometrical paradigms and Spaces for Geometric Work that articulates two basic views on a geometer's work: cognitive and epistemological.

Keywords Geometric work · Visualisation · Geometrical paradigm

Introduction

The influence of tools, especially drawing tools, on Geometry development at school has recently improved greatly due to the appearance of DGS. The traditional opposition between practical and theoretical aspects of geometry has to be rethought. It's well known that we can approach Geometry through two main routes: AQ1

1. A concrete approach which tends to reduce geometry to a set of spatial and practical knowledge based on material world.
2. An abstract approach oriented towards well organized discursive reasoning and logical thinking.

K. Alain (✉)
Laboratoire de Didactique André Revuz, Université Paris Diderot, Paris, France
e-mail: alain.kuzniak@univ-paris-diderot.fr

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With the social cynicism of the Bourgeoisie in the mid-nineteenth century, the first approach was for a long time reserved to children coming from the lower class and the second was introduced to train the elite who needed to think and manage society.

Today, in France, this conflict between both approaches stays more hidden in Mathematics Education but such discussions have reappeared with the social expectation supported by the Organisation for Economic Co-operation and Development (OECD) and its "bras armé" Programme for International Student Assessment (PISA) with the opposition between "Mathematical literacy" and "Advanced Mathematics".

In the present paper, I will leave aside sociological and ideological aspects and focus on what could be a didactic approach, keeping in mind a possible scientific approach to a more practical geometry referring to approximation and measure, in the sense Klein used when he suggested a kind of approximated Pascal's theorem on conics:

> Let six points be roughly located on a conic: if we draw the lines roughly joining points and they intersect at a, b and c, then these points are roughly aligned. (Klein 1903).

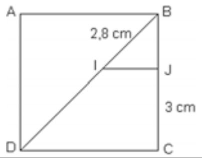
The present presentation will be supported by a first example showing what kind of contradiction exists in French Education where no specific work on approximation exists during compulsory school. This contradiction appears as a source of confusion and misunderstandings between teachers and students. We were lead to introduce some theoretical perspectives aiming at understanding and solving this trouble. In the following, our theoretical framework for studies in Geometry will be introduced and used to launch some perspectives.

Complexity of the Geometric Work

Mathematical domains are constituted by the aggregation and organization of knowledge. A mathematical domain is the object of various interpretations when it is transformed to be taught. These interpretations will also depend on school institutions. The case of geometry is especially complex at the end of compulsory school, as we will show in the following.

The following problem was given for the French examination at Grade 9 in 1991 (Table 1).

Table 1 A geometric problem

Construct a square ABCD with side 5 cm 1. Compute BD 2. Draw the point I on [BD] such that BI = 2.8 cm, and then the point J on [BC] such that JC = 3 cm Is the line (IJ) parallel to the line (DC)?	

The intuitive evidence (the lines are parallel) contradicts the conclusion expected from a reasoning based on properties (the lines are not parallel). Students are faced with a variety of tasks referring to different, somewhat contradictory conceptions and the whole forms a fuzzy landscape:

1. In the first question, a real drawing is requested. Students need to use some drawing and measure tools to build the square and control and validate the construction.
2. Students then have to compute a length BD using the Pythagorean theorem and not measure it with drawing tools. But which is the nature of the numbers students have to use to give the result: An exact value with square roots, or an approximate one with decimal numbers which is well adapted to using constructions and that allows students to check the result on the drawing?
3. In the third question—are the lines parallel?—students work again with constructions and have to place two points (I and J) by measuring lengths. Moreover, giving the value 2.8 can suggest that the length is known up to one digit and could encourage students to use approximated numbers rounded to one digit. In that case is equal to 1.4 and both ratios are equal, which implies the parallelism by the Thales' Theorem related to similarity. If students keep exact values and know that is irrational, the same Theorem implies that the lines are not parallel.

With Grade 9 Students

The problem was given in a Grade 9 class (22 students), one week after a lecture on exact value with square roots and its relationships to length measurement. After they had spent 30 min working on the problem, half of the students answered that the lines were parallel and the other half answered that they were not. On the teacher's request, they used the problem of approximated values to explain the differences among them. At the teacher's invitation, they started again to think about their solutions. At the end, 12 concluded the lines were not parallel, 8 that they were and 2 hesitated.

Indeed, after studying their solutions and their comments on the problem, we can conclude that students' difficulties did not generally relate to a lack of knowledge on geometric properties, but to their interpretations of the results. They had trouble with the conclusions to be drawn from Thales' Theorem. Even after discussion, students expressed their perplexity about the result and its fluctuation. One student said "I don't know if they are parallel for when I round off, the ratios are equal and so the lines are parallel, but they are not parallel when I take the exact values". For students, one answer is not more adequate than another. This gives birth to a geometric conception where some properties could be sometimes true or false. How to make students overcome the contradiction? A first possibility is to force the entrance in the didactical contract expected by the class's teacher, who explained that at this moment in Grade 9, it must be clear that "a figure is not a proof".

Working on approximation and thinking about the nature of geometry taught during compulsory school open a second way we will explore with geometrical paradigms in the following.

Geometrical Paradigms and Three Elementary Geometries

The previous example and numerous others of the same kind show that a single viewpoint on geometry would miss the complexity of the geometric work, due to different meanings that depend both on the evolution of mathematics and school institutions. At the same time, we saw that students are strongly disturbed by this diversity of approaches. Geometrical paradigms were introduced into the field of didactics of geometry to take into account the diversity of points of view (Kuzniak and Houdement 1999, 2003). AQ2

The idea of geometrical paradigms was inspired by the notion of paradigm introduced by Kuhn (1962, 1966) in his work on the structure of the scientific revolutions. In a global view, one paradigm consists off all the beliefs, techniques and values shared by a scientific group. It indicates the correct way for putting and starting the resolution of a problem. Within the restricted frame of the teaching and learning of geometry, our study is limited to elementary geometry, and the notion of paradigm is used to pinpoint the relationships between geometry and belief or mathematical theories.

With the notion of paradigms, Kuhn has enlarged the idea of a theory to include the members of a community who share a common theory.

> A paradigm is what the members of a scientific community share, and, a scientific community consists of men who share a paradigm (Kuhn 1966, p. 180).

When people share the same paradigm, they can communicate very easily and in an unambiguous way. By contrast, when they stay in different paradigms, misunderstandings are frequent and can lead, in certain cases, to a total lack of comprehension. For instance, the use and meaning of figures in geometry depend on the paradigm. Sometimes it's forbidden to use the drawing to prove a property by measuring and only heuristic uses of figures are allowed.

To bring out geometrical paradigms, we used three viewpoints: epistemological, historical and didactical. That led us to consider the three following paradigms described below.

Geometry I: Natural Geometry

Natural Geometry has the real and sensible world as a source of validation. In this Geometry, an assertion is supported using arguments based upon experiment and deduction. Little distinction is made between model and reality and all arguments are allowed to justify an assertion and convince others of its correctness. Assertions are proven by moving back and forth between the model and the real: The most important thing is to develop convincing arguments. Proofs could lean on drawings or observations made with common measurement and drawing tools such as rulers, compasses and protractors. Folding or cutting the drawing to obtain visual proofs are also allowed. The development of this geometry was historically motivated by practical problems.

The perspective of Geometry I is of a technological nature.

Geometry II: Natural Axiomatic Geometry

Geometry II, whose archetype is classic Euclidean Geometry, is built on a model that approaches reality. Once the axioms are set up, proofs have to be developed within the system of axioms to be valid. The system of axioms could be incomplete and partial: The axiomatic process is a work in progress with modelling as its perspective. In this geometry, objects such as figures exist only by their definition even if this definition is often based on some characteristics of real and existing objects.

Both Geometries have a close link to the real world even if it is in different ways.

Geometry III: Formal Axiomatic Geometry

To these two approaches, it is necessary to add a third Geometry (Formal Axiomatic Geometry) which is little present in compulsory schooling but which is the implicit reference of teachers' trainers when they have studied mathematics in university, which is very influenced by this formal and logical approach.

In Geometry III, the system of axioms itself, disconnected from reality, is central. The system of axioms is complete and unconcerned with any possible applications in the world. It is more concerned with logical problems and tends to complete "intuitive" axioms without any "call in" to perceptive evidence such as convexity or betweenness. Moreover, axioms are organized in families which structure geometrical properties: affine, euclidean, projective, etc.

These three approaches (and this is one original aspect of our viewpoint) are not ranked: Their perspectives are different and so the nature and the handling of problems change from one to the next. More than the name, what is important here is the idea of three different approaches of geometry: Geometry I, II and III.

Back to the Example

If we look again at our example, students—and teachers—are not explicitly aware of the existence of two geometrical approaches to the problem, each coherent and possible. And students generally think within the paradigm which seems natural to them and close to perception and instrumentation—Geometry I. But in this geometry, measurement is approximated and known only over an interval. Parallelism of lines depends on the degree of approximation. Teachers insist on a logical approach—Geometry II—which leads the students to conclude blindly that the lines are not parallel, against what they see.

It could be interesting to follow Klein's ideas and introduce a kind of "approximated" theorems, more specifically here an "approximated" Thales' Theorem: If the ratios are "approximately" equal then the lines are "almost" parallel. In that case, it would be possible to reconcile what is seen on the drawing and what is deduced based on properties.

Developing thinking on approximation in Geometry can be supported by DGS which favour a geometric work into Geometry I but with a better control of the degree of approximation. It is the case, for instance, with the CABRI version we used during the session with students. In this version, an "oracle" is available which can confirm or not the validity of a property seen on the drawing. Here, the parallelism of the two lines was confirmed by the "oracle" according to the approach with approximation of the problem.

Many problems allow discussion of the validity of a theorem or property in relationship to numerical fields. For instance, CABRI oracle asserts that (EF) and (BC) are parallel lines in a triangle ABC when E and F are respectively defined as the midpoints of [AB] and [AC]. But, if E is defined as the midpoint of [AB], when we drag a point F on [AC] it is possible that CABRI oracle never concludes that (EF) and (BC) are parallel for any position of F. These variations in the conclusion need an explanation and provoke a discussion among students which can be enriched by the different perspectives on Geometry introduced by geometrical paradigms.

To discuss the question in-depth and think about new routes in the teaching and learning of geometry, we will introduce some details about the notion of Space for Geometric Work.

The Notion of Space for Geometric Work Within the Framework of Didactics of Geometry

At school, Geometry is not a disembodied set of properties and objects reduced to signs manipulated by formal systems: It is at first and mainly a human activity. Considering mathematics as a social activity that depends on the human brain leads to understanding how a community of people and individuals use geometrical paradigms in everyday practice of the discipline. When specialists are trying to

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solve geometric problems, they go back and forth between the paradigms and they use figures in various ways, sometimes as a source of knowledge and, at least for a while, as a source of validation of some properties. However, they always know the exact status of their hypotheses and the confidence they can give to each one of these conclusions.

When students do the same task, we are not sure about their ability to use knowledge and techniques related to Geometry. That requires an observation of geometric practices set up in a school frame, and, more generally, in professional and everyday contexts, if we aim to know common uses of mathematics tools. The whole work will be summarized under the notion of *Space for Geometric Work* (SGW), a place organized to enable the work of people solving geometric problems. Individuals can be experts (the mathematician) or students or senior students in mathematics. Problems are not a part of the Work Space but they justify and motivate it.

Architects define Work Spaces as places built to ensure the best practice of a specific work (Lautier 1999). To conceive a Work Space, Lautier suggests thinking of it according to three main issues: a material device, an organization left at the designers' responsibility and finally a representation which takes into account the way the users integrate this space. We do not intend to take up this structure oriented to the productive work without any modifications, but it seems to us necessary to keep in mind these various dimensions, some more material and the others intellectual.

The Epistemological Level

To define the Space for Geometric Work, we introduced three characteristic components of the geometrical activity into its purely mathematical dimension. These three interacting components are the following:

A real and local space as material support with a set of concrete and tangible objects.
A set of artefacts such as drawing instruments or software.
A theoretical frame of reference based on definitions and properties.

These components are not simply juxtaposed but must be organized with a precise goal depending on the mathematical domain in its epistemological dimension. This justifies the name *epistemological plane* given to this first level. In our theoretical frame, the notion of paradigms brings together the components of this epistemological plane. The components are interpreted through the reference paradigm and in return, through their different functions, the components specify each paradigm. When a community can agree on one paradigm, they can then formulate problems and organize their solutions by favouring tools or thought styles described in what we name the reference SGW. To know this SGW, it will be necessary to bring these styles out by describing the geometrical work with rhetoric rules of discourse, treatment and presentation.

The Cognitive Level

We introduced a second level, centred on the cognitive articulation of the SGW components, to understand how groups, and also particular individuals, use and appropriate the geometrical knowledge in their practice of the domain. From Duval (2005), we adapted the idea of three cognitive processes involved in geometrical activity.

A visualization process connected to the representation of space and material support;
A construction process determined by instruments (rules, compass, etc.) and geometrical configurations;
A discursive process which conveys argumentation and proofs.

From Gonseth (1945–1952), we retained the idea of conceiving geometry as the synthesis between different modes of knowledge: intuition, experiment and deduction (Houdement and Kuzniak 1999).

The real space will be connected to visualization by intuition, artefacts to construction by experiment and the reference model to the notion of proof by deduction. This can be summarized in the following diagram (Fig. 1). AQ3

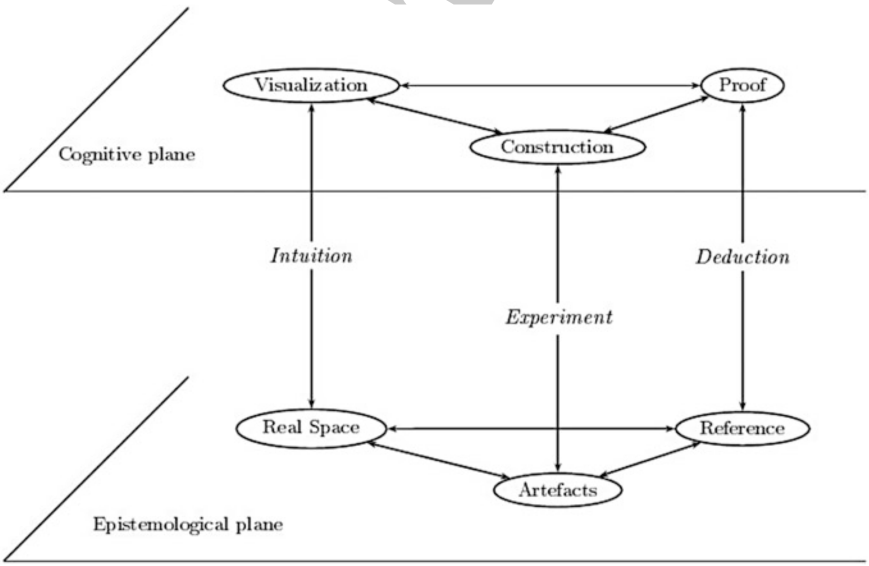

Fig. 1 The space for geometric work

Building a Space for Geometric Work: A Transformation Process

On the Meaning of Genesis

In the following, we will consider the formation of SGW by teachers and students. Our approach intends to better understand the creation and development of all components and levels existing in the diagram above. The geometric work will be considered as a process involving creation, development and transformation. The whole process will be studied through the notion of genesis, used in a general meaning which is not only focused on origin but also on development and transformation of interactions. The transformation process takes place and, finally, forms a structured space, the Space for Geometric Work.

Various SGW Levels

In a particular school institution, the resolution of geometric tasks implies that one specific SGW has been developed and well organized to allow students to enter into the problem solving process. This SGW has been named appropriately and the appropriate SGW needs to meet two conditions: it enables the user to solve the problem within the right geometrical paradigm, and it is well built, in the sense in which its various components are organized in a valid way. The designers play a role similar to architects conceiving a working place for prospective users. When the problem is put to an actual individual (young student, student or teacher), the problem will be treated in what we have named a *personal SGW*. The geometric work at school can be described thanks to three SGW levels: Geometry intended by the institution is described in the reference SGW, which must be fitted out in an appropriate SGW, enabling an actual implementation in a classroom where every student works within his or her personal SGW.

Various Geneses of the Space for Geometric Work

As we have seen, geometrical work is framed through the progressive implementation of various SGW. Each SGW, and specifically the personal SGW, requires a general genesis which will lean on particular geneses connecting the components and cognitive processes essential to the functioning of the whole Geometric Working Space. The SGW epistemological plane needs to be structured and organized through a process oriented by geometrical paradigms and mathematical considerations. This process has been named *epistemological genesis*. In the same way, the cognitive plane needs a cognitive genesis when it is used by a generic or

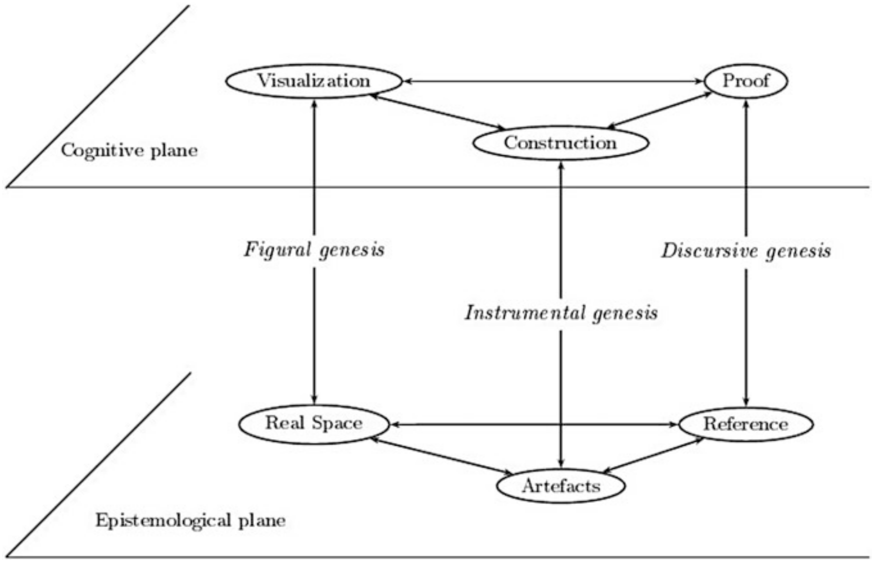

Fig. 2 Geneses into the geometrical work space

particular individual. Specific attention is due for some cognitive processes such as visualization, construction and discursive reasoning.

Both levels, cognitive and epistemological, need to be articulated in order to ensure a coherent and complete geometric work. This process supposes some transformations that can be pinpointed through three fundamental geneses strictly related to our first diagram (Fig. 2):

An *instrumental* genesis which transforms artefacts in tools within the construction process.

A *figural* and *semiotic* genesis which provides the tangible objects their status of operating mathematical objects.

A *discursive* genesis of proof which gives a meaning to properties used within mathematical reasoning.

We will examine how it comes into geometrical work by clarifying each genesis involved into the process.

On Figural Genesis

The visualization question came back recently to the foreground of concerns in mathematics and didactics after a long period of ostracism and exclusion for suspicion.

In geometry, figures are the visual supports favoured by geometrical work. This led us, in a slightly restrictive way, to introduce a figural genesis within the SGW framework to describe the semiotic process associated with visual thinking and involved in geometry. This process has been especially studied by Duval (2005) and Richard (2004). Duval has given some perspectives to describe the transition from a drawing seen as a tangible object to the figure conceived as a generic and abstract object. For instance, he spoke of a biologist viewpoint when it is enough to recognize and classify geometric objects such as triangle or Thales' configurations often drawn in a prototypical way. He also introduced the idea of dimensional deconstruction to explain the visual work required on a figure to guide the perceptive process. In that case, a figure needs to be seen as a 2D-object (a square as an area), a set of 1D-objects (sides) or 0D-objects (vertices). Conversely, Richard insists on the coming down process from the abstract and general object to a particular drawing.

On Instrumental Genesis

A viewpoint on traditional drawing and measuring instruments depends on geometrical paradigms. These instruments are usually used for verifying or illustrating some properties of the studied objects. The appearance of computers has completely renewed the question of the role of instruments in mathematics by facilitating their use and offering the possibility of dynamic proofs. This aspect is related to the question of proof mentioned in the preceding paragraph, but the ability to drag elements adds a procedural dimension which further increases the strength of proof in contrast to static perception engaged in paper and pencil environments. But the ability with the use of artefacts is not easy to reach by the students. At the same time, teachers need to develop specific knowledge for implementing software in a classroom. Based on Rabardel's works on ergonomic, Artigue (2002) stressed the necessity of an instrumental genesis with two main phases that we can insert in our frame. The coming up transition, from the artefacts to the construction of geometric configurations, is called instrumentation and gives information on how users manipulate and master the drawing tools. The coming down process, from the configuration to the adequate choice and the correct use of one instrument, related to geometric construction procedures, is called instrumentalisation. In this second process, geometric knowledge are engaged and developed.

On Discursive Genesis of Reasoning

The geometrization process, which combines geometric shapes and mathematical concepts, is central to mathematical understanding. We saw the strength of images or experiments in developing or reinforcing certainty in the validity of an

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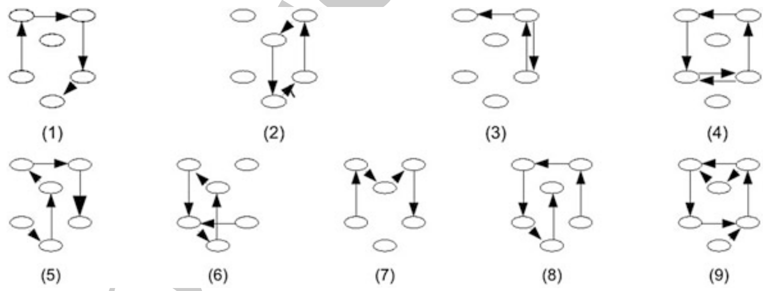
announced result. However, how can we make sure that students understand the logic of proof when they do not express their argumentation in words, but instead base it on visual reconstructions that can create illusions? A discursive explanation with words is necessary to argue and to convince others.

The nature and importance of written formulations differ from one paradigm to another. In most axiomatic approaches, it is possible to say that mathematical objects exist only in and by their definition. This is obviously not the case in the empiricist approach, where mathematical objects are formed from a direct access to more or less prototypical concrete objects.

Towards a Coherent Geometric Work at the End of Compulsory School

Using the theoretical framework introduced above, we will insist here on some contradictory ways we encountered in French geometry education and highlight what could be a coherent approach using both geometric paradigms. For that, we draw some conclusions from a work of Lebot (2011) who has studied different ways of teaching the introduction for the notion of angles at Grades 6 to 8. Using the SGW diagram, it is possible to describe possible routes students may take when they use software or drawing tools to construct figures and solve problems. Lebot has observed interesting differences visible on the following diagrams and we will discuss some among them.



A Coherent GI Work Space

Generally, a geometric task begins with a construction performed using either traditional drawing tools or digital geometric software. Each time, the construction is adjusted and controlled by the gesture and vision.

In this approach to geometry, the trail into the SGW diagram is like the one of Diagram 5 and done in a first sense (Instrumental—Figural and then Discursive) which characterizes an empirical view on geometric concepts.

A coherent way to work theoretically in Geometry I would be to use "approximated" theorems in the sense we introduced (Section "A Coherent GII Work Space") where the numerical domain is based on decimal numbers rather than real numbers. Theoretical discourse must justify what we see and not contradict it. This approach has been developed by Hjelmsev (1939) among others. AQ4

A Coherent GII Work Space

In the Geometry II conception, the focus is first on the discourse that structures the figure and controls its construction. This time, the route is trailed (Diagram 8) in an opposite sense (Discursive—Figural—Instrumental) and the figure rests on its definition: All properties could be derived from the definition without surprises.

In the traditional teaching and learning of geometry, students are frequently asked to start geometric problems with the construction of real objects. This leads them to work in the sense (I-F-D) of the Diagram 5. But for the teacher, the actual construction of an object is not really important. The discursive approach is preferred and expected, as in the Diagram 8 covered in sense (D-F-I): what I know is stronger than what I see and measure. In this pedagogical approach, elements coming from Geometry I support students' intuition for working in Geometry II, leading the formation of a (GII/GI) Work Space. But at the same time, students may believe that they work in a (GI/GII) Work Space where the objective is to think about real objects using some properties coming from Geometry II (Thales and Pythagorean Theorems) to avoid direct measurement on the drawing. The geometric work made by students could be incomplete as in Diagram 6 where students stay in an experimental approach without any discursive conclusion. They have paid attention to the construction task which requires time and care, but this work is neglected in the proof process expected by the teacher, where figures play only a heuristic supporting role. That can lead to another form of incomplete work but this time favoured by teachers as in Diagram 4 where there exists only interaction between proof and figure.

The inverse circulation of the geometric work in Geometry I and Geometry II can lead to a break in the geometric work that forms, when only one approach is explicitly privileged. We support the idea that both geometric paradigms must be included in geometry learning to develop a coherent (G|GII) Work Space where both paradigms have the same importance. Only when this condition is met, can an approximation have both a numerical and geometrical meaning, and can a work space be created suitable for introducing "almost parallel" lines in relationship to decimal numbers and where "strictly parallel" relate to real numbers. That would help resolve problems of mathematical coherency such as those experienced by students who asserted that they did not know if the lines were parallel because “the lines (IJ) and (DC) are parallel if we round off, but they are not if we take the exact value”.

Beyond the Space for Geometric Work

How can the notion of SGW be extended beyond the Geometry? First, we can take into account the context within which the geometric work is developed. This context can be of social nature or could deal with the cognitive dimension in the teaching and learning processes as Arzarello and Robutti (2008) did by introducing the “Space of Action, Production and Communication” viewed as metaphorical space where the student’s cognitive processes mature through a variety of social interactions. Within these frameworks, it is clear that the notion of SGW can operate and pinpoint on what, at the end, is the goal of an educational approach in mathematics: to make an adequate mathematical work. This assertion leads us to another kind of generalization related to what is mathematical work. In this direction, we have started some investigations with researchers interested in Calculus, Probability or Algebra. A third symposium on this topic has been held in Montreal in 2012 and some elements on this approach are given in Kuzniak (2011). The generalization supposes an epistemological study in-depth of the specific mathematical domain and of its relationships to other domains. Indeed, each domain relates to a particular class of problems and the crucial question is to find an equivalent to the role that space has in geometry. Variations and functions for calculus, chance and data for probability and statistics, can play the same role as space and figures in geometry. If it seems that the two planes, epistemological and cognitive, keep the same importance as in geometric work, figural genesis and vizualisation should be changed and reinterpreted through semiotic and representation processes in relationship to the mathematical domain concerned. But it is another story and work in progress.

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