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How do teachers' approaches to geometric work relate to geometry students' learning difficulties?

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Abstract Various studies suggest that French students (grades 7 to 10) may solve geometric problems within a paradigmatic framework that differs from that assumed by teachers, a situation prone to misunderstandings. In this paper, we study the extent to which secondary school teachers recognise the conflicting paradigms and how they handle the geometric work conducted, in sometimes unintended ways, by their students. This is done by analysing teachers' reactions to specific answers students offered to the Charlotte and Marie problem, an "ambiguous" problem with various solutions depending on the paradigm adopted. As a result of the study, we found that, beyond similarities due to a shared mathematical background, the way secondary schoolteachers handle students' answers varies with their conceptions of geometric work. Implications are drawn regarding the teaching of geometry and the training of teachers.

Keywords Geometry · Geometrical paradigms · Geometric work · Epistemic conception

1 The general purpose

1.1 On knowledge required for teaching mathematics

Since the early 1970s, numerous studies carried out in several countries have evaluated teachers' mathematics subject knowledge based upon the common belief that the greater the subject knowledge, the better the teaching. In the case of geometry, mathematical subject knowledge among teachers appears uneven with a lot of gaps especially among elementary school teachers. Yet, the link between limitations in subject knowledge and the quality of the teaching is not easy to study. Identifying the influence of these teachers' difficulties on their practice and ultimately on the performance of their students is particularly problematic. As Ball, Lubienski & Mewborn (2001) pointed out, very few studies have focussed on this question and existing ones have sometimes provided some paradoxical

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results, as in Begle's paper (1979, cited by Ball et al. 2001, p. 442) in which he concluded that greater mathematics subject knowledge could be associated with a negative effect on students' achievement.

Given these limits, research on mathematics teacher education conducted during the 1980s was focussed on what teachers really need to teach in the classroom. Shulman (1986) introduced the notion of pedagogical content knowledge (PCK) to complement subject content knowledge, and based on this idea, various refinements have been made to describe knowledge that is really needed to teach mathematics. Recently, Hill, Ball & Schilling (2008) have introduced the notion of knowledge of contents and students (KCS) and of knowledge of content and teaching (KCT) to organise mathematical knowledge for teaching (MKT). Working from the didactic research perspective, the related notion of didactical content knowledge (Houdement & Kuzniak, 1996, 2001) was similarly introduced for the purpose of investigating the knowledge of the didactics of mathematics teachers needed in the classroom.

Kahan, Cooper & Bethea (2003) studied the link between mathematics teachers' knowledge and the lessons they teach. They tested 16 students in both elementary geometry and algebra, taking into consideration the particular mathematics syllabi in use. The trainee teachers then prepared lesson plans, which were evaluated by one of the researchers. Although they focussed on mathematical content knowledge (MCK), the authors noted in their conclusion (p. 248) that "our analysis repeatedly invoked pedagogical content knowledge (PCK) and affective considerations to explain surprising results".

To describe geometric knowledge and teaching geometric knowledge, Chinnappan & Lawson (2005) used the notion of concept mapping introduced into science education by Novak (1990). They drew the geometric knowledge maps of two teachers by using graphs consisting of nodes and labelled lines. The nodes were used to indicate concepts while the lines showed how two distinct concepts were related. They did not draw any conclusions about the relationship between subject knowledge and teaching.

In another approach, Steinbring (1998) insisted on the necessity of working on epistemological knowledge to improve teaching and proposed to transform current practices by changing teachers' attitudes and understanding of mathematics. This is the approach we adopted in this paper, using the notion of geometrical paradigms in an effort to explain some of the difficulties encountered in the teaching and learning of geometry. In so doing, we intend to assist students and teachers in overcoming some of the recurrent misunderstandings among them in geometry.

1.2 Identifying the sources of misunderstandings

Suspecting that among French secondary schoolteachers, who generally have an adequate level in mathematics, the sources of difficulties are of a didactical or epistemological nature, we will use the notions of obstacle and misunderstanding to define our research objectives. Based on Bachelard's (1938) notion of epistemological obstacle, Brousseau (1997) proposed that the teaching and learning of mathematics could be hampered by three kinds of obstacles distinguished by their origins: ontogenic, epistemological, or didactical. The first two terms refer to students' abilities or understanding, and the last one relates more to the choices teachers make within an educational system. Whereas epistemological obstacles "are those from which one neither can nor should escape, because of their formative role in the knowledge being sought," (Brousseau, 1997, p. 87) didactical obstacles may arise as a result of teachers' specific choices of methods or because of their limited knowledge of their students' cognitive capacities.

Problems of a didactical nature may also occur when students do not do geometric work exactly as teachers expect them to, even if they think this work valid. To study these kinds of inconsistency, especially in both their social and cognitive dimensions, the concept of misunderstanding will be used. Specifically, misunderstanding occurs when students' views of what they need to do deviate from their teachers' implicit expectations. For instance, mathematics teachers insist on the need to learn and know theorems, which their students subsequently learn by heart, but for a teacher fully knowing a theorem necessarily implies that the students know how to apply it. In practice, when learning geometry, students spend a lot of time drawing precise geometric figures that are no more than visual supports for the teacher. Brousseau insists on the fact that the source of and the solution to these misunderstandings could depend on the didactical contract (Brousseau, 1997) developed in the classroom.

1.3 The teaching of geometry

In geometry education, the notions of geometrical paradigms (Houdement & Kuzniak, 1999, 2003) and of geometric work spaces (Kuzniak, 2006) have been used to clarify the different meanings of the term geometry. Using those tools, we formulated the following propositions, which our work abundantly supports.

> In education, three distinct paradigms structure the field of geometry. These paradigms reflect various stages in the succession of academic cycles. Each stage is characterised by specific practices and challenges in the teaching and learning of the discipline.

We labelled these three paradigms Geometry I, II and III, two of which—Geometry I and II—play an important role in today's secondary education. Each paradigm is global and coherent enough to define and structure geometry as a discipline and to set up respective work spaces suitable for solving a wide range of problems.

This first proposition was complemented by the following one about the potential impact of these paradigms on geometry education.

> Students and teachers may adopt implicitly different paradigms; differences in approaches may lead to misunderstandings.

Thus, the geometric principles organised and taught by teachers could be interpreted differently from what the teacher intends and this can lead to conflicting standpoints where teachers and students misinterpret or mischaracterise their respective intents and performances.

This article focusses on teachers' epistemological conceptions and how they analyse and deal with the differences across paradigms and the particular or even unorthodox geometry work done by their students. We shall relate it to their epistemological approach to geometry. In the next section we shall introduce the theoretical framework used for our investigation.

2 Theoretical framework and questions for an analysis of the nature of teachers' work on geometry

2.1 Notions of geometrical paradigms and geometric work space

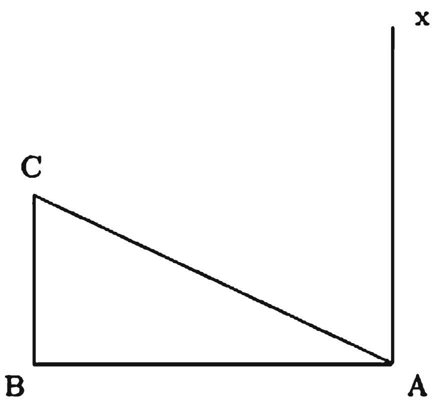
The theoretical framework taking into account geometrical paradigms and geometric work spaces has been developed with two broad considerations. First, we needed to describe and explain the challenges encountered in the teaching of geometry in the French education system at the end of compulsory schooling (grades 8–10) when the contents of geometry education change. Second, we had to develop an overall analytical framework that could be used in a comparative study with the Chilean system where the choices made by policy makers were clearly contrasting with the French approach.

2.1.1 An example

The presentation of the theoretical framework starts with one geometric problem selected to display the content of geometric work from the perspective of the various paradigms during compulsory geometry education. The example chosen is taken from the competitive mathematical examinations that French students must pass to become primary schoolteachers (teaching children from age 3 to age 11). These problems usually assume basic elementary mathematical knowledge and are equivalent to Grade 9 level.

2.1.2 Problem

Let ABC be a triangle with a right angle in B, with AB=4 cm and BC=2 cm. The ray (Ax) is perpendicular to the line (AB). And M is a point on the ray (Ax). The purpose of this problem is to obtain particular configurations of the triangle AMC.



Question: Does a point M exist such that the triangle ACM is equilateral? Justify your reply

The correct answer is "no" and it can be shown, using a compass, that there is no third vertex on the ray (Ax) for the equilateral triangle constructed on the side (AC).

Such a response is emblematic of Geometry I. A student carries out an experiment in the real, perceptible world by constructing a triangle with drawing instruments and then s/he realises that no crossing points lie on the line where one point should be for the triangle to be equilateral. The argument is supported by diagrams, objects that are typical and central to Geometry I.

This response, however, is rejected in French traditional education at this grade level: Pre-service teachers and students above grade 8 must avoid a solution such as this in order to pass their examination. In the French curriculum, students are required to argue and justify their solutions without using the diagram and in many textbooks (level 8 and upwards), formulating a proof on a basis of a diagram is ruled out.

Instead, answers must be based on written deductive proof supported by hypotheses and geometric properties, and not by illustration with diagrams.

> For example, if ACM is an equilateral triangle with M on Ax, the angle MAC will measure 60° and the angle CAB 30° (sum of the three angles of a triangle) and by symmetry the angle CAC' will be 60° (C' is the symmetric of C through the line (AB)). As the triangle CAC' is isosceles in A (by symmetry), it should be equilateral. This is not true because the length of C'C is 4, which is unequal to CA and C'A (2.sqrt(5) by Pythagoras' theorem).

The latter solution is illustrative of Geometry II. A reasoned deductive argument is constructed on the basis of initial data and geometric theorems. The diagram does not fulfil the same function as in Geometry I: in Geometry II, it is used for its heuristic value and not as a vehicle of proof.

The example highlights two main points.

1. There exist various viewpoints on elementary geometry, which can result in conflicting solutions to the same problem, as we shall see later in the "Charlotte and Marie" problem (Sect. 3.1)
2. Certain terms (such as construction) can assume different meanings depending on teachers' and students' standpoints on the same question

The notion of geometrical paradigm is useful to understand, clarify and organise these various points of view.

2.2 Geometrical paradigms and geometric work spaces

2.2.1 The notion of paradigm according to Kuhn

Kuhn introduced the notion of paradigms in his fundamental work about scientific revolutions (1962, 1966). In this book, Kuhn used this term many times and after some approximations, he defined it by insisting on two aspects.

In its most global use, the term paradigm stands for the entire constellation of beliefs, values, techniques, practices etc. shared by the members of a given community. In this sense, Kuhn speaks about *disciplinary matrix*, a term that refers to the cultural, symbolic and metaphysical background common to the practitioners of a discipline.

More specifically, the term paradigm refers to one element in that constellation, i.e. the concrete puzzle solutions that, employed as models or examples, can replace explicit rules as the basis for the solution of the remaining puzzle of normal science.

The concept of paradigm broadens the notion of theory and relates it to the existence of a community of individuals who share a common theory. A paradigm is what the members of a scientific community share, and, a scientific community consists of men who share a paradigm (Kuhn, 1966, p. 180).

When people share the same paradigm, they can communicate very easily and in an unambiguous way. On the contrary, when they stay in different paradigms, misunderstandings are frequent and can lead, in some cases, to a total lack of comprehension. For instance, how diagrams are used in geometry depends on the paradigm adopted: using them to prove a property by measuring is sometimes excluded.

2.2.2 Three elementary geometries

Following the epistemological viewpoint of Gonseth (1945–1955) who posits geometry in relation to the problem of space, three geometrical paradigms are considered in order to organise the interplay among intuition, deduction and reasoning in relation to space:

1. Geometry I finds its validation in the material and tangible world; therefore, its name *natural geometry*. In this geometry, valid assertions are generated using arguments based upon perception, experiment and deduction. The confusion between the model and reality is great and any argument is allowed to justify an assertion and to convince the reader. Indeed, dynamic and experimental proofs are acceptable in Geometry I. In essence, the perspective of Geometry I is technological.
2. Geometry II (natural axiomatic geometry), whose archetype is classic Euclidean geometry, is built on a model that approaches reality without being fused with it. Once the axioms are set up, proof has to be developed within the system of axioms to be valid. Axioms are closely associated with our perception of space, hence the name *natural axiomatic geometry*. The system of axioms may be left incomplete as the axiomatic process is dynamic and has modelling at its core.
3. Both geometries have close links to the real world albeit in varying ways. In particular, they differ with regard to the type of validation, the nature of the figure (unique and specific in Geometry I, general and definition-based in Geometry II) and by their work horizons. To these two approaches, it is necessary to add Geometry III (*formal axiomatic geometry*), which is little present in compulsory schooling, but which is the implicit reference of mathematics teachers who are university trained. In Geometry III, the system of axioms itself, disconnected from reality, is central. The system is complete and unconcerned with any possible applications to the real world. The connection with space is severed and this geometry is more concerned with logical problems.

These various paradigms—and this is an original feature of our approach—are not organised in a hierarchy making one more preferable than another. Their work horizons are different and the choice of a path towards the solution is determined by the purpose of the problem and the investigator’s viewpoint.

2.2.3 The notion of geometric work space

Geometry, and more generally mathematics, as taught in school, is a human activity that is embedded in a social system and cannot be reduced to abstract signs managed by formal systems. Considering mathematics as a social activity that is carried out by a human brain can help us understand how communities and individuals adopt one geometric paradigm or another in the day to day practice of the discipline. When specialists try to solve some geometric problems, they go back and forth between paradigms, they may use diagrams for various purposes, sometimes as objects of study, and sometimes, if only temporarily, as a means of validation of some properties. However, they always know the exact status of their hypotheses and the confidence they can give to each of these hypotheses.

A new series of questions with regard to geometry’s users arises when we think about geometry as human work. This work depends on the role given to visualisation and drawing instruments in the validation process. It also depends on the pattern of properties and definitions of geometric objects. Ultimately, it depends on the personal belief and knowledge of the individual student.

Therefore, in order to describe the complexity of geometric work, the notion of geometric work space (GWS) was introduced. The GWS (Kuzniak, 2006) is a place that is organised to ensure the work of people solving geometry problems (e.g. geometricians). It establishes the reference to the complex setting in which the problem solver acts. It comprises two planes that are at the same time material and intellectual: the components plane and the cognitive plane.

In the components plane, three elements intersect:

1. The real and local space as material support with a set of concrete objects
2. Artefacts such as drawing instruments and software available
3. A theoretical system of reference consisting of definitions and properties.

By themselves, the components are not sufficient to define the global meaning of a GWS, which depends on the function that its designer and its users gave to it. A first reorganisation of these various components will be epistemological in nature and directed by geometrical paradigms.

The cognitive plane was introduced to describe the cognitive activity of a particular user and Duval's approach was used to clarify the cognitive processes involved in problem solving in geometry. Adapting Duval (1995), three processes were defined:

1. A visualisation process with regard to space representation and the material support
2. A construction process determined by the instruments used (ruler, compass etc.) and the geometric configuration
3. A discursive reasoning process that conveys argumentation and proof.

2.2.4 Relations between GWS and geometrical paradigms

A GWS exists only through its users, current or potential. Its constitution depends on the way users combine these two planes and their components for solving geometric problems. It also depends on the cognitive abilities of a particular user, expert or beginner.

The make-up of a GWS will vary with the education system (the intended GWS), the school circumstances (the implemented GWS) and on the practitioners (students' and teachers' personal GWS). In practice, the constitution of a GWS does not rely on a single paradigm, but rather on the interplay among different paradigms. We will illustrate this fact by presenting some possible interconnections among the paradigms that we encountered in the comparative studies we conducted. This description is always made at the level of the intended GWS at the end of compulsory education and it generally differs from the two others. Each type of GWS was labelled according to its corresponding paradigm.

Established Geometry I The study of configurations from the real world is the goal of Geometry I (Guzman & Kuzniak, 2006), where it is allowed and even encouraged to take measurements on a figure to solve problems. Some theorems can also be used as technical tools to replace measurement by calculations: this is the case for Pythagoras' Theorem or the Intercept Theorem (Thales' Theorem in France).

Established Geometry II Geometry II (Vivier & Kuzniak, 2009) rests on a set of properties and experiments provided by Geometry I and is intuitively useful. But then the axiomatic horizon is clearly a part of this geometry, which brings it close to Euclidean geometry.

Fragmented Geometry II As in the previous case, fragmented Geometry II is based on a set of properties and experiments issued from Geometry I. But in contrast to the previous case, this geometry is characterised by discrete blocks of hypothetico-deductive reasoning organised around properties and some basic geometric configurations. These blocks of reasoning are founded on a few properties justified by an experiment validated by measurement or by software.

2.3 Some questions about the nature of the teachers' geometric work space

Within our theoretical framework, we can clarify our investigation and focus on the following three questions:

1. What paradigm do the teachers intend to focus on through the teaching of geometry: Geometry I or Geometry II or an articulation between the two paradigms?
2. How do they approach the various components of the GWS and how much importance do they give to each component? In particular, what place do they give to the various components (artefacts, properties, space and figure) and to the cognitive processes (visualisation and construction in relation to discursive reasoning)?
3. Do teachers take into account the way students understand geometry and their difficulties tackling problems, and if so, how? For instance, do they understand that the ambiguous status of the drawing could cause students to falter and err, and in that case, how do they manage the difficulty?

With the first two questions, we investigate teachers' epistemological conceptions about the geometry they teach. With the second question, we furthermore address their didactic conception about the building of an appropriate GWS for the teaching of geometry by focussing on the components plan (relationships between the real and local space, artefacts and reference system). With the last question, we seek to examine the notion of GWS with regard to learning by focussing on the cognitive plane (the role of methods using perception and visualisation, various methods of proof and the significance of students' difficulties).

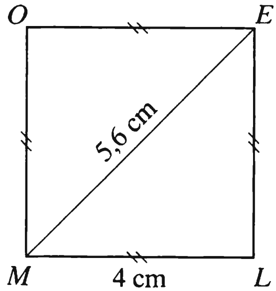
3 Methodology

To answer our questions and gain a better view of teachers' GWS, secondary school-teachers were invited to complete a questionnaire, presenting them with answers that students gave to the "Charlotte and Marie" geometric problem and that were collected in a former study (Rauscher & Kuzniak, 2005; Kuzniak, 2008).

The problem—Sect. 3.1—serves our purpose because it yields different solutions depending on the approach adopted by the solver. It is considered ambiguous in our theoretical frame because it is possible to solve it in different ways based on different conceptions of geometry. Students' answers to this problem vary considerably: not only do they reflect different approaches, but they also show differences in skill levels in the performance of geometric work. In our former study, we found that four types of answers could be distinguished—Sect. 3.2—. The four typical answers were submitted to participating teachers as part of the questionnaire—Sect. 3.3.

3.1 An ambiguous problem (P): Charlotte and Marie

The study is based on the following problem, which will be discussed in detail below. For geometric problems of this type, defining a work space suitable for solving the problem may be challenging, especially for the second question. Depending on the viewpoint on the figure, different solutions orientated by Geometry I or II are possible.



Question: Why can we assert that the quadrilateral OELM is a rhombus? Marie maintains that OELM is a square. Charlotte is sure that it is not true. Does a point M exist such that the triangle ACM is equilateral? Justify your reply.

Who is right?

The diagram looks like it is showing a square, but its status in the problem is not clear. Is the drawing a *real object*? Is it the actual object of study posited by the question or is it only an illustration resulting from a textual description? In this latter case, is the material representation of the figure necessary or is it used only as a heuristic support for reasoning? The function of the depicted object is usually stated in the wording of the problem. Once established, the function determines the adoption of a precise geometrical paradigm. In the exercise above, the wording gives no such indications and one student did legitimately point out: "There is no descriptive text in the wording, but just a drawing that can mislead."

Finally, how should the question be answered and who, out of Charlotte and Marie, is right? The answer depends on the paradigm selected, as we will show with two coherent solutions using Pythagoras' theorem. This theorem, which does not require an actual measurement of the angle, provides a typical method for solving this kind of question.

From the standpoint of Geometry II, it is possible to argue that Charlotte is right, as one student did:

> We know that if OEM has a right-angle in O then we have $OE^2+OM^2=ME^2$
>
> We verify that $4^2+4^2=5.6^2$ and 32 is not equal to 31.26. Thus, OEM is not a right-angled triangle.

If Geometry I is the work horizon, the reasoning proposed by another student leads to a different conclusion: Marie is right; OELM is a square, since sqrt(32) is approximately equal to 5.6.

In fact, it would be necessary to conclude that OELM is "almost" a square with the help of a specific and practical form of Pythagoras' theorem that relies on approximate numbers and, in a less common way, approximate figures.

> The triangle ABC is "almost" right in B if, and only if, AB^2+BC^2 is approximately equal to AC^2.

In this case, the direct and converse forms of the theorem are not separated.

Thus, two coherent solutions are possible, depending on which paradigm is adopted. The first one leads to work within Geometry II; it breaks away from the sensible experimental data and reasons in the numeric domain. The second formulation is associated with Geometry I, albeit an advanced form of it. Nevertheless, for lack of adequate language in the curriculum, students and teachers cannot play on these distinctions within the discipline of geometry. They are faced with a potential misunderstanding that is both epistemological and didactic, and what teachers think about these contradictions merits a more detailed investigation.

3.2 Typical students' answers used to study teachers' reactions

The "Charlotte and Marie" problem came from a textbook designed for 14-year-old students and, as mentioned above, it is particularly ambiguous. This problem was given to primary schoolteachers and to grade 9 students. A detailed analysis and classification of these answers appear in Rauscher & Kuzniak (2005) and Kuzniak (2008). In the present study, characteristic answers were presented to secondary schoolteachers in an attempt to uncover how they approached geometry and how they understood students' answers. What follows is a summary of Kuzniak's (2008) results. Some students' answers are given in the Appendix.

Four categories of answers to the "Charlotte and Marie" problem can be identified, each labelled PII, PIprop*erty,* PIperc*eption*, and Piexp*eriment*.

Answers using Pythagoras' theorem are common to PII and PIprop. Students who did not use Pythagoras' theorem were categorised as Plexp and PIperc.

3.2.1 PII

In this case, the standard Pythagoras' theorem is applied inside the world of abstract figures and numbers without considering the actual aspect of the object. Only information given by words and signals (i.e. coding of equal segments, lengths) is used, and Pythagoras' theorem is applied in its full formal rigour. To prove that the quadrilateral is a rhombus (four sides of the same length) and to show that it is not a square (contrapositive to Pythagoras' theorem), students use minimal and sufficient properties. The GWS of this population is shaped by Geometry II.

3.2.2 PIprop

This category groups together students who applied the practical Pythagoras' theorem, indeed, its converse. In general, they concluded that Marie is right. In this case, students recognise the importance of the drawing and measurement approximation. The practical Pythagoras' theorem appears as a tool of Geometry I. This category of students was labelled PIprop to underscore the fact that its members reason using properties. The question is whether these students can master the interplay between Geometry I and Geometry II or whether their horizons remain purely technological in Geometry I.

3.2.3 Plexp

Students who used their measuring and drawing tools to arrive at an answer were placed in a separate group situated in the experimental world of Geometry I. Generally, students in this category concluded that Marie was right. However, that was not always the case: one student, using his compass, verified that the vertices of the quadrilateral were not co-cyclic and thus could assert that OELM is not a square.

3.2.4 PIperc

In this last category, students whose answers were based on perception were grouped together: their interpretation of the drawing was the basis for their answer, and they did not give us any information about their tools of investigation. It is not easy to know whether this lack of deductive proof is due to a lack of geometric knowledge or simply to absolute reliance on their perception of the diagram. To answer this question, we must consider the difficulties these students encounter when reasoning.

3.3 The questionnaire given to the teachers

With the methodology developed by Rauscher (1993, 1994), our study uses responses to questions submitted to 20 secondary school teachers during a training course in Strasbourg. The session was conducted by the authors and was designed to introduce teachers to the paradigm approach. These in-service teachers had volunteered to attend this course. Half of them were teaching in "Collèges" (Grades 6–9) and the other half in "Lycées" (Grades 10–12). They had all followed the same initial course of studies in mathematics.

The secondary school teachers were first asked to respond to a questionnaire using the "Charlotte and Marie" problem as a means of exposing their own conceptions about geometry and to bring out what they thought about their students' solutions. After solving the two questions in the exercise, the teachers were asked to respond in writing to the following questions:

1. What doubts and confusion may students have experienced in this exercise?
2. What difficulties and confusion may students have experienced in this exercise?

With these questions, we wanted to see how the teachers dealt with: the ambiguity of the problem, gaps in students' knowledge, and their own teaching.

Then teachers were asked to comment on seven answers from students. The answers were selected on the basis of our categorisation and reflected a wide variety of solutions. Some answers with specific mistakes were selected because they showed the use of non-characteristic properties and confusion between properties.

Teachers were asked to react to the following questions:

1. Which student's solution is the closest to the solution you would give in class?
2. How would you propose to address the questions that students' answers might raise?

The first question was given in closed form to force teachers to express their expectations about the students' work. The second question was kept open, leaving teachers free to express how they manage students' answers and difficulties.

4 Analysis of the teachers' answers

4.1 Four comparison points guiding the analysis

The questions posed to the teachers gave much insight into their geometric knowledge and teaching methods. But, clearly, a single observation will not reflect the totality of a teacher's stance on geometry and its teaching. Yet, looking at the whole set of answers makes it possible to describe that stance and teachers' personal GWS. Our analysis was focussed on four comparison points in attempting to expose the multiplicity of GWS.

4.1.1 The written solution to the problem

We assessed the validity of the proofs (a priori in Geometry I or in Geometry II) that teachers gave to the "Charlotte and Marie" problem. These answers provided indications of how teachers mastered the system of reference within the paradigm they used to solve the exercise.

4.1.2 The relationships between Geometry I and Geometry II

We sought to find out whether teachers were implicitly aware of the existence of different solutions to the problem depending on the choice of paradigm. Did they only offer a solution according to Geometry II or did they think of alternative solutions, possibly conceived within Geometry I? Did any of them consider the problem to be ambiguous?

Did any teacher approach the problem looking at the drawing as an object of Geometry I and a heuristic support for reasoning in Geometry II, and if so, how did he/she express it, if at all?

4.1.3 Were students' difficulties a concern for teachers?

What, according to teachers, causes students to experience difficulties? Did the teachers insist on adopting formal rules and principles? Or did their answers reflect an intrinsic difficulty in the understanding of Geometry II, which rests on real objects used as empirical supports for abstractions within Geometry II.

4.1.4 Teaching recommendations to help struggling students

When teachers pinpoint the causes of difficulties in the confusing interplay between paradigms, did they offer any ideas or recommendations to overcome them?

We conducted an analysis of the teachers' answers making extensive use of our framework of paradigms, even when teachers referred to them only implicitly. The first step was to group together answers showing a similar approach to a particular question. From the analysis of teachers' answers according to the four points listed above, similarities and differences observed in the organisation of each teacher's GWS (Sect. 2.3) were recorded to evaluate the following:

1. The paradigm intended by the teachers
2. Their approach to the relationships between the various components of the GWS and the interplay between paradigms
3. Their consideration of the students' level of understanding and methods.

We will present the results of the study by illustrating our synthesis with excerpts of teachers' responses.

4.2 An overview of the similarities and differences between the teachers in our sample

Whereas teachers showed remarkable homogeneity and conformity as to the format given to solutions, they demonstrated considerable divergence with regard to the other points. There emerged deep differences in profiles and teaching routines, reflecting sharp contrasts between approaches to geometry. The following presents our analysis of each point in turn.

4.2.1 A great homogeneity on the geometrical paradigm intended by the teachers

In this study, teachers universally worked and reasoned within the Geometry II paradigm and established with their students a rapport within Geometry II as shown in the following observations.

We noted firstly that teachers, in solving the problem, had a good command of geometric knowledge; specifically, they all used the hypotheses given in the problem and applied definitions or theorems correctly to reach rigorous answers in Geometry II. In contrast to earlier studies with primary schoolteachers (Rauscher & Kuzniak 2005), they all concluded that Charlotte was right after using Pythagoras' theorem. Two of them nevertheless made a critical analysis of the problem and questioned the reference paradigm (Geometry I or II) used to decide between Charlotte and Marie.

Numerous details were given by the teachers on the degree of rigour they expected in such proof writing. Some of them insisted on rigorous written proof and required that only the direct form of Pythagoras' theorem (a proof using a contrapositive) was to be used and not its reciprocal form (although one teacher mistakenly confused the two forms). They underlined the difficulties students experienced in developing such formal and rigorous reasoning.

Several teachers stressed the misleading role of the drawing. They pointed out that students could *incorrectly* try to solve the problem in Geometry I by relying on the appearance of the drawing or applying an approximate Pythagoras' theorem (Sect. 3) and thus assert that Marie was right. A teacher summarised the idea by speaking of a *visual trap*. A junior high school teacher (grades 6 to 9) wrote:

> Students run the risk of calculating ME by means of the Pythagoras' theorem and finding that the triangle is almost a right triangle.

In this way, these teachers underlined the fact that a misunderstanding can occur between students solving the problem within Geometry I and the teacher expecting a solution within Geometry II.

4.2.2 Teachers promoting only a theoretical approach or taking into account an empirical view of geometry

Beyond a common perspective dictated by Geometry II, teachers differed in their awareness of the ambiguousness of the problem and the role of the drawing.

Almost all the teachers emphasised how difficult, but necessary it was for students to learn and apply the relevant tools (theorems or definitions) for solving the problem in Geometry II. One teacher (grades 6 to 9) wrote about the difficulties that students can encounter when solving the problem:

> Mastery of theorems of geometry (rhombus+one right angle=square), mastery of Pythagoras' theorem; mastery of definitions of squares and rhombus.

Some teachers evoked only the theoretical dimension (properties and theorems) of the GWS and they did not pay attention to the other components, in particular to the role of the real and local space and the instruments.

In contrast to this clear-cut vision of geometry, other teachers suggested that difficulties could arise from the actual drawing as we have seen above.

Thus, a senior high school teacher (grades 10 to 12) wrote:

> From the way the figure is drawn, many students see a square. They do not pay attention to the coding.

Even if a majority of answers made no reference to the drawing as a visual support, some comments referred to the need to use it in order to locate a specific point, for example, if "OEM is a right-angled triangle, it can only be at O". On the other hand, other teachers said nothing about these questions and gave a classical answer as directed by Geometry II.

As seen above, the reference to Geometry I is generally viewed as a mistake, but a few teachers analysed the ambiguity of the problem, which could just as well be considered a problem that could be solved within Geometry I:

One teacher (grades 9 to 10) contrasted questions 1 and 2

1. It is a rhombus because its four sides are all the same length. (It is easy and clearly specified on the figure).

2. Now it gets hard. It is enough to show that ELM is a right angle with Pythagoras' theorem. But, is the square root of 32 equal to 5.6? I would say both (Charlotte and Marie) are right but for different reasons.

Similarly, one teacher (grades 10 to 12) wrote:

> We do not know who is right (Charlotte or Marie) since the length of EM is given as 5.6 cm, i.e. as an approximate value and not an exact one (but a carpenter would be satisfied with his work if he had made a window with that degree of precision).

These teachers were referring to another approach to geometry, one that depends on measurement and approximation and not strictly on the application of a theoretical framework.

4.2.3 In organising the GWS, teachers differed markedly in the way they looked at the methods used by students

When we analysed teachers' responses regarding the difficulties experienced by students, we found that differences between teachers increased and our initial categorization was confirmed.

Some teachers did not attempt to distinguish between the elements of Geometry I and Geometry II in the solutions advanced by students. These teachers expected solutions within Geometry II only and emphasised the obligation to present a rigorous, written demonstration. Let us consider the position held by Etienne, a junior high school teacher (grades 6 to 9). He talked at length about several difficulties experienced by students, but he made no reference to the actual diagram, which itself could misdirect or generate uncertainties. He only stressed the formal writing he expected:

> I asked the students to write down the hypotheses, to define the property explicitly in quotes and then to write their conclusions.

Consistent with this approach, he stressed the importance of standard mathematical definitions and properties that students would be expected to know and use.

He suggested that "it might be necessary to revise and correct any misunderstandings relating to the properties of squares and rhombuses" and continued as follows:

> The first question would not cause any trouble. A rhombus is defined as a quadrilateral with four equal sides. This definition has already been taught and should have been understood in Grade 6.

However, in contrast to this strict view of students' obligations, other teachers drew attention to working within the geometric framework of Geometry II while striving to reason on objects from the familiar universe of Geometry I. One teacher (grades 6 to 9), who pointed out that it was necessary to look at the diagram in order to locate the right angle, insisted that it was difficult for students to disregard the visual evidence provided by the diagram in their reasoning:

> In question 2, it is true, that can readily be seen on the figure. But why and how can it be justified? How might we define a rhombus as distinct from a square?

Did the teachers have ideas or suggestions on how to deal with students' questions? Here, the differences among the teachers increased. Teachers like Etienne who sought to encourage his students to write formal proofs in their answers, said very little about this. Etienne only wrote that the problem "could be very instructive for his students to try to solve: it is *incontournable*—"it cannot be ignored" in basic French. He presumed that students could use this exercise to achieve the rigour expected in Geometry II.

By contrast, among teachers aware of the potential problems that may arise during the students' transition from Geometry I to Geometry II, some teachers seemed to be at a loss, not knowing what to do, while others suggested potential solutions for managing the transition from one paradigm to another. Typical of the problems shown by those who fall into the first group, Gerard, a junior high school teacher, underscored the problem that he experienced adding an exclamation mark.

> My problem: making it clear to the students that the figure is not a square despite the visual evidence!

However, he did not give any potential solutions. Another teacher, working in a senior high school reiterated only the rules of formal reasoning:

> Remember that we have to prove what we assert. It has not been proven that the two diagonals have the same midpoint. Discuss the principles of causes and consequences and then what derives from what? A theorem and its converse.

In the latter group, some teachers offered ideas and suggestions that could have led their students to question the status of the diagrams in Geometry I and prepare the transition to Geometry II. Thus, Arthur, a senior high-school teacher (grades 10 to 12), gave this reply to the question on how to run the class:

> Work on what is necessary and ensure that your explanation is sufficient by drawing diagrams that verify certain propositions and then ask questions about the nature of the final diagram.

He commented on the example given.

> I can see it so it must be true, but then after making this deduction I need to follow up by writing a simple proof that may show that my deduction may be incorrect.

He also drew attention to the need for a clear understanding of geometric concepts.

Working on the exact and the approximate values of a square root.

We can see in this line of reasoning potential solutions leading to coherent teaching that takes into account the analysis of the interplay between Geometry I and II.

5 General conclusion and prospective issues

We shall summarise our research questions and the results of the study and outline some implications for research and teachers' training.

5.1 Some contrasting attitudes reflecting how the teachers take into account students' difficulties

We studied the reactions of teachers when faced with characteristic difficulties encountered by students trying to solve a problem. Given our theoretical framework, the problem was selected to reveal specific difficulties that emerge with the change of paradigms at the end of French compulsory school. Our purpose was to determine whether teachers were aware of the ambiguous character of the problem and to analyse their reaction to it.

Beyond a common horizon dictated by Geometry II, we noticed differences between teachers' attitudes and their teaching methods. Among the teachers, some remained within a GWS only directed by Geometry II and did not consider Geometry I as a possible source of geometric notions and challenges for students. When they referred to Geometry I objects, it was only to assert that these objects were not consistent with the model. These teachers worked within the confines of formal proof with their students.

On the other hand, some teachers were implicitly aware of the two paradigms and the fact that they shaped their students' personal GWS. They offered perspicacious analyses of the tensions created by the different geometries and a few of these teachers suggested teaching techniques designed to remedy and overcome these tensions.

Most teachers were to some degree aware of the existence of paradigms. It would be interesting to study to what extent this variation in teaching practices depends on the institution where they teach. In our sample, teachers exhibited different profiles even when teaching in the same institution and even if junior high school teachers often insisted on adopting the formal proof method, a method that was introduced initially in this institution. Surprisingly, senior high school teachers appeared less adamant about adopting a formal approach and showed more flexibility in their teaching of geometry. We cannot, however, draw any conclusions as to whether this result depends on our sample or is more general.

The study focussed on some specific elements of the geometric work space: the role of diagrams in relation to the type of validation. It will be necessary to go further in the analysis of the geometric work by examining the place of theorems in the teaching process from the standpoint of geometry teachers, as Herbst & Miyakama (2008) did.

Teachers seem to look at students' performance through some preconceived notions of geometry, which leads them to adopt different attitudes with respect to the difficulties they encounter. More generally, it would be interesting to cross this result with their conceptions of the learning process and to determine and compare both epistemological and educational influences.

5.2 Some implications for the teacher training

Our study reinforces today's current notion in mathematics education that task treatment and problem resolution vary with the context in which they are applied. In his PhD thesis, Rauscher (1993, 1994) presented evidence that there is a large "teacher effect" that affects students' performance, taking into account social and cognitive differences across classes. In this paper, we focussed on the differences among teachers, i.e. their various conceptions about what constitutes legitimate geometric work and what significance a teacher should assign to students' uncertain dealings with real and ideal figures.

Our long-term project is to affect teaching by changing teachers' attitudes towards mathematics and their teaching. However, the road that changes habits is long and the rigidity of teachers' activities in the classroom is great (Pardies, Robert & Rogalski, 2008) and to be effective various approaches have to be developed. As we mentioned in the introduction, our study intends to contribute to the field in which researchers such as Steinbring (1998) insisted on the necessity of working on epistemological knowledge to improve teaching. We think that a description in terms of GWS teachers' profiles could make the sources of rigidity in teaching better understood and help open new ways of teacher training. We saw that teachers' personal GWS dictates how they perceive their students' solutions; consequently, changing teachers' views on their students' work and on teaching will require an in-depth transformation of their work space.

We have started to work towards this end by giving teachers some epistemological tools that can help them understand their students' difficulties and quickly adjust their teaching methods. The notion of geometrical paradigms is useful in bringing out teachers' preconceptions and enabling them to change their teaching as their didactical content knowledge increases.

The approach we have applied can be represented by a chain reaction and summarized as follows:

(P) Problem→	(S) Students' answers to (P)→	(SR) *Students' reaction to (S)*
	↓	
	(T) Teachers' reactions to (S)	
	↓	
	(TR) *Teachers' (or teachers' trainers) reactions to (T)*	

This method is expected to help teachers recognise diversity among both their students and their colleagues. Because of their cultural bias with regard to their profession, French teachers generally work alone with minimum exchange with their colleagues, concentrating on their classes and ignoring other practitioners, which partly explains the relative complexity of the study. French teachers are not in the habit of communicating with their colleagues and sharing good practice.

It is also possible to work directly with secondary school students (level SR) or pre-service primary school teachers (Rauscher & Kuzniak, 2005; Parzysz & Jore, 2009). In this paper we have described a part of the process through analysing the level (T). We will use this analysis in the future to build the next step (TR) and to evaluate the impact of this process on teaching practices.

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Appendix: students' answers

PII

1. OELM is a rhombus since its successive sides are equal.
2. If OELM is a square, then MEL is a right-angled triangle at L. According to Pythagoras' theorem we would have then, $ME^2=ML^2+LE^2$. As $ML^2+LE^2=16+16=32$ and $ME^2=5.6^2=31.36$ thus, the angle ELM is not right.

Consequently, OELM is not a square and it is Charlotte who is right.

PIprop

1. OELM is a rhombus, for OE=OM=M L=LE and a rhombus has four sides of the same length.
2. Marie is right since all the sides of the quadrangle have the same length and there is at least one right angle. We can verify it by Pythagoras' theorem.

$ME^2=ML^2+LE^2$ $4^2+4^2=16+16=32$
ME=Sqrt(32) approximately equal to 5.6 thus MEL =90°.

PIexp

1. OELM is a rhombus, for its diagonals cut each other at their midpoint (measuring) by forming right angles (using a set square). *Remark: the student has drawn the second diagonal on the figure.*
2. Marie is right. It is a square, for besides being a rhombus, OELM has right angles (set square).

PIperc

1. The opposite sides of the quadrilateral are parallel and with the same length OE=ML and OM=EL. According to the definition of a rhombus, we can say that diagonals have the same midpoint and are perpendicular.
2. Marie is right; OELM is also a square since its sides form a right angle.

References

Bachelard, G. (1938). *La formation de l'esprit scientifique.* Paris: Vrin. (Translation *Formation of the Scientific Mind.* Clinamen Press).

Ball, D. L., Lubienski, S. T., & Mewborn, D. S. (2001). Research on teaching mathematics: the unsolved problem of teacher's mathematics knowledge. In Richardson, V. (Ed.), *Handbook of research on teaching*, 4th ed (pp. 433–456). Washington, DC: American Educational Research on Education.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht: Kluwer.

Chinnappan, M., & Lawson, M. J. (2005). A framework for analysis of teachers' geometric content knowledge and geometric knowledge for teaching. *Journal of Mathematics Teacher Education, 8*(3), 197–221.

Duval, R. (1995) Why to teach geometry? *ICMI Study on Geometry* (pp. 53–58). Catania.

Gonseth F. (1945–1955). *La géométrie et le problème de l'espace [Geometry and the Space Problem]*. Lausanne: Editions Le Griffon.

Guzman, I., & Kuzniak, A. (2006). *Paradigmes géométriques et géométrie enseignée au Chili et en France [Geometrical Paradigms and Geometry taught in Chile and France]*. Paris: Irem Paris-Diderot.

Herbst, P., & Miyakawa, T. (2008). When, how, and why prove theorems? A methodology for studying the perspective of Geometry teachers. *ZDM—The International Journal on Mathematics Education, 40*(3), 469–486.
Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372–400.
Houdement, C., & Kuzniak, A. (1996). Autour des stratégies utilisées pour former les maîtres du premier degré en mathématiques [About strategies used to train primary school teachers in mathematics]. *Recherches en Didactique des Mathématiques, 16*(3), 289–322.
Houdement, C., & Kuzniak, A. (1999). Un exemple de cadre conceptuel pour l'étude de l'enseignement de la géométrie en formation des maîtres [A Conceptual Framework to study Geometry Teaching in Teachers Training]. *Educational Studies in Mathematics, 40*(3), 283–312.
Houdement, C., & Kuzniak, A. (2001). *Pretty (good) didactical provocation as a tool for teachers' training in Geometry. Proceedings of CERME2* (pp. 292–304). Praha: Charles University.
Houdement, C., & Kuzniak, A. (2003). Elementary Geometry split into different geometrical paradigms. *Proceedings of CERME 3.* Bellaria, Italy.
Kahan, J. A., Cooper, D. A., & Bethea, K. A. (2003). The role of mathematics teacher's content knowledge in their teaching: a framework for research applied to a study of students teachers. *Journal of Mathematics Teacher Education, 6*(3), 223–252.
Kuhn, T.S. (1962). *The structure of scientific revolutions.* Chicago: University of Chicago Press.
Kuhn, T.S. (1966). *The structure of scientific revolutions,*, 2nd ed. Chicago: University of Chicago Press.
Kuzniak, A. (2006). Paradigmes et espaces de travail géométriques [Paradigms and Geometric Work Spaces]. *Canadian Journal of Science and Mathematics, 6*(2), 167–187.
Kuzniak, A. (2008). Personal geometrical working space: a didactic and statistical approach. In Gras, R., Suzuki, E., Guillet, F., Spagnolo, F. (Eds) *Statistical implicative analysis: theory and applications studies in computational intelligence,* vol. 12. Heidelberg: Springer.
Novak, J. (1990). Concept mapping: a useful device for science education. *Journal of Research in Science Education, 27*, 937–949.
Pardies, M., Robert, A., & Rogalski, J. (2008). Analyses de séances en classe et stabilité des pratiques d'enseignants de mathématiques expérimentés du second degré [Analysis of Teaching and Secondary School Teachers Stability of Practices]. *Educational Studies in Mathematics, 68*(1), 55–80.
Parzysz, B., & Jore, F. (2009). Studying French preservice elementary teachers' relation to Geometry through their discourse. In Winslow, C. (Ed.), *Nordic research in mathematics education* (pp. 117–126). Rotterdam: Sense Publishers.
Rauscher, J.-C. (1993). *L'hétérogénéité des professeurs face à des élèves hétérogènes. Le cas de l'enseignement de la géométrie au début du collège [Heterogeneous teachers faced to heterogeneous students. The case of Geometry Teaching at the beginning of High School]*. Strasbourg: University of Strasbourg.
Rauscher, J.-C. (1994). Les enjeux de l'enseignement de la géométrie au début du collège et leur prise en compte par les professeurs [Geometry Teaching Stakes at the beginning of High School and Teachers Approaches]. In Artigue, M., Gras, R., Laborde, C., Tavignot, P. (Eds.), *Vingt Ans de Didactique des Mathématiques en France, Recherches en Didactique des Mathématiques* (pp. 19–198). Grenoble: La pensée sauvage.
Rauscher, J-C. & Kuzniak, A. (2005). On geometrical thinking of pre-service school teachers. *Proceedings of CERME4* (pp. 738–747). Sant Feliu de Guíxols, Spain.
Shulman, L. S. (1986). Those who understand: knowledge growth in teaching. *Educational Researcher, 15* (2), 4–14.
Steinbring, H. (1998). Elements of epistemological knowledge for mathematics teachers. *Journal of Mathematic Teacher Education, 1*(2), 157–189.
Vivier, L., & Kuzniak, A. (2009). A French look on the Greek Geometrical Working Space. *Proceedings of CERME6.* Lyon, France.